Proofs Homework Set 3

MATH 217 — WINTER 2011

Due January 26

A few words about proofs. This is our first set of proof problems. There will be additional proof problems accompanying every assignment for the rest of the semester. Here are some suggestions to keep in mind:

Write down your solutions in full, as if you were writing them for another student in the class to read and understand.

Don't be sloppy, since your solutions will be judged on precision and completeness and not merely on"basically getting it right."

Cite every theorem or fact that you are using. ("By Theorem 1.10 ..." , or "By the theorem from class which states that for every matrix such that ... we also have ...", etc.)

If you compute something by observation, say so and make sure that your fellow imaginary student (who is reading your proof) can also clearly see what you are claiming.

Justify each step in writing and leave nothing to imagination.

PROBLEM 3.1. In this problem, we will completely determine which 2×2 matrices give which reduced echelon forms. For now on, set

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

for $a, b, c, d \in \mathbb{R}$. We assume throughout that at least one of a, b, c, d is nonzero, i.e., that M is *not* the 2×2 matrix with all entries zero.

(a) Give an example of an M where a, b, c, d are all nonzero and $ad - bc = 0$. Show that your M has reduced echelon form

$$\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$$

for some $y \neq 0$.

Proof. For example, let $a = b = c = d = 1$. Then $ad - bc = 1 \cdot 1 - 1 \cdot 1 = 0$. The resulting matrix $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ has reduced echelon form $\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ (*i.e.*, $y = 1$). □

(b) Generalize your computation from part a to prove that any M satisfying that a, b, c, d are all nonzero and $ad - bc = 0$ has reduced echelon form

$$\begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}.$$

(Note that dividing by a is okay, since we assumed $a \neq 0$.)

HINT. Start off by multiplying the top row by d, which is legal since $d \neq 0$, and then the bottom row by $-b$, which is again okay, since $b \neq 0$.

Proof. Following the instructions in the hint, we see that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} ad & bd \\ c & d \end{bmatrix} \sim \begin{bmatrix} ad & bd \\ -bc & -bd \end{bmatrix}.$$

Then we add the top row to the bottom row to find

$$\begin{bmatrix} ad & bd \\ -bc & -bd \end{bmatrix} \sim \begin{bmatrix} ad & bd \\ ad - bc & bd - bd \end{bmatrix} = \begin{bmatrix} ad & bd \\ 0 & 0 \end{bmatrix}.$$

Finally, to get the reduced echelon form, we simply divide the top row by ad (which is not zero) to obtain $\begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}$. $\square$

(c) Suppose that $ad - bc = 0$ at least one of a, b, c, d is zero. Prove that M has at least one column or row with all zeros.

Proof. We consider four different cases.

Suppose first that $a = 0$. Then $-bc = ad - bc = 0$, which means that one of b and c is zero. If $b = 0$ then the top row is zero; if $c = 0$ then the left column is zero.

Suppose now that $b = 0$. Then $ad = ad - bc = 0$, which means that one of a and d is zero. If $a = 0$ then the top row is zero; if $d = 0$ then the right column is zero.

Suppose next that $c = 0$. Then $ad = ad - bc = 0$, which means that one of a and d is zero. If $a = 0$ then the left column is zero; if $d = 0$ then the bottom row is zero.

Suppose finally that $d = 0$. Then $-bc = ad - bc = 0$, which means that one of b and c is zero. If $b = 0$ then the right column is zero; if $c = 0$ then the bottom row is zero. $\square$

(d) Show that if M has exactly one column with all zeros, then the reduced echelon form of M is one of

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Proof. Suppose the left column is zero. Then $M = \begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix}$, where at least one of b and d is nonzero. If b is nonzero, we can rescale the top row by $1/b$ and subtract d times the resulting row to the bottom row to find $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & d \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. If d is nonzero, we can rescale bottom row by $1/d$ and subtract b times the resulting row to the top row to find $\begin{bmatrix} 0 & b \\ 0 & d \end{bmatrix} \sim \begin{bmatrix} 0 & b \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Suppose the right column is zero. Then $M = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}$, where at least one of a and c is nonzero. In a manner similar to the above, we find that $M \sim \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. □

(e) Show that if M has exactly one row with all zeros, then the reduced echelon form of M is one of

$$\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

where y might be zero.

Proof. Suppose the bottom row is zero. Then $M = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$, where at least one of a and b is nonzero. If $a \neq 0$, then we can rescale the top row by $1/a$ to find $M \sim \begin{bmatrix} 1 & b/a \\ 0 & 0 \end{bmatrix}$; if $a = 0$ then $b \neq 0$ and we can rescale the top row by $1/b$ to find $M \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Suppose the top row is zero. Then $M = \begin{bmatrix} 0 & 0 \\ c & d \end{bmatrix}$, where at least one of c and d is nonzero. If $c \neq 0$, then we can rescale the bottom row by $1/c$ to find that $M \sim \begin{bmatrix} 0 & 0 \\ 1 & d/c \end{bmatrix} \sim \begin{bmatrix} 1 & d/c \\ 0 & 0 \end{bmatrix}$; if $c = 0$ then $d \neq 0$ and we can rescale the bottom row by $1/d$ to find $M \sim \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. □

(f) Suppose that $ad - bc \neq 0$. Prove that M has reduced echelon form the 2×2 **identity matrix**:

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

HINT. Start off with the same row operations as in part b, add one row to another, then show that the first column is a pivot column. Then, do some more row operations to show that the second column is also a pivot column.

Proof. First note that b and d cannot both be zero; at least one must be nonzero.

If $d \neq 0$ then we find that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} ad & bd \\ c & d \end{bmatrix} \sim \begin{bmatrix} ad - bc & 0 \\ c & d \end{bmatrix},$$

where the last matrix was obtained by subtracting b times the second row to the first. Since $ad - bc \neq 0$, we can divide the top row by $ad - bc$ to obtain

$$\begin{bmatrix} ad - bc & 0 \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix}.$$

We can then subtract c times the first row from the second row to obtain

$$\begin{bmatrix} 1 & 0 \\ c & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & d \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

where the last rescaling of the second row was made possible by the fact that $d \neq 0$.

If $b \neq 0$ then we find that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \sim \begin{bmatrix} a & b \\ -bc & -bd \end{bmatrix} \sim \begin{bmatrix} a & b \\ ad - bc & 0 \end{bmatrix},$$

where the last matrix was obtained by adding d times the first row to the second row. Since $ad - bc \neq 0$, we can divide the second row by $ad - bc$ to find

$$\begin{bmatrix} a & b \\ ad - bc & 0 \end{bmatrix} \sim \begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix}.$$

We can then subtract a times the second row to the first row to find

$$\begin{bmatrix} a & b \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & b \\ 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

where the last rescaling of the second row was made possible by the fact that $b \neq 0$. Exchanging the two rows now gives the desired result. $\square$

Part (f) is in fact an if and only if statement. That is:

THEOREM. *A* 2×2 *matrix*

$$M = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

with $a, b, c, d \in \mathbb{R}$*, has reduced echelon form the* 2×2 *identity matrix if and only if* $ad - bc \neq 0$.

Proof. By part (f), we know that if $ad - bc \neq 0$ then M has reduced echelon form $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

Now suppose $ad - bc = 0$. We consider three cases; in each case we will find that the reduced echelon form of M is not the 2×2 identity matrix.

Case a, b, c, d are all nonzero. In this case, part (b) tells us that M has reduced echelon form $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ where $y \neq 0$; this is not the 2×2 identity matrix.

Case a, b, c, d are all zero. In this case, $M = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is already in reduced echelon form and it is not the 2×2 identity matrix.

Case at least one but not all of a, b, c, d are zero. In this case, part (c) tells us that M has at least one zero column or one zero row, which leads to two subcases:

- In the subcase when M has one zero column, then part (d) tells us that the reduced echelon form of M is one of $\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, neither of which is the 2×2 identity matrix.
- In the subcase when M has one zero row, then part (e) tells us that the reduced echelon form of M is one of $\begin{bmatrix} 1 & y \\ 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$, neither of which is the 2×2 identity matrix. $\square$