Proofs Homework Set 5

MATH 217 — WINTER 2011

Due February 9

PROBLEM 5.1. If A and B are $n \times n$ matrices which are row equivalent, prove that AC and BC are row equivalent for every $n \times n$ matrix C. We will do this in two parts.

(a) Show that if $A \sim B$ (that is, if they are row equivalent), then $EA = B$ for some matrix E which is a product of elementary matrices.

Proof. By definition, if $A \sim B$, there is some sequence of elementary row operations which, when performed on A, produce B. Further, multiplying on the left by the corresponding elementary matrix is the same as performing that row operation. So we have

$$A \sim E_1 A \sim E_2 E_1 A \sim \cdots \sim E_p E_{p-1} \cdots E_2 E_1 A = B$$

Thus, if $E = E_p E_{p-1} \cdots E_2 E_1$, we have $EA = B$. □

(b) Show that if $EA = B$ for some matrix E which is a product of elementary matrices, then $AC \sim BC$ for every $n \times n$ matrix C.

Proof. Write $E = E_p E_{p-1} \cdots E_2 E_1$ where each E_i is an elementary matrix. Then

$$AC \sim E_1 AC \sim E_2 E_1 AC \sim \cdots \sim E_p E_{p-1} \cdots E_2 E_1 AC = EAC$$

Since $EA = B$, we can multiply on the right by C to get $EAC = BC$. Therefore $AC \sim BC$, as claimed. □

PROBLEM 5.2. An **upper triangular matrix** is a square matrix in which the entries below the diagonal are all zero, that is, $a_{ij} = 0$ whenever $i > j$. An example is the 4×4 matrix $\begin{bmatrix} 4 & 5 & 10 & 1 \\ 0 & 7 & -1 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 9 \end{bmatrix}$.

Let A be a $n \times n$ upper triangular matrix with nonzero diagonal entries. In this problem, you will build up the pieces necessary to prove that A is invertible and that the inverse of A is also upper triangular.

(a) Prove that the elementary matrices corresponding to the row operations of

- scaling, and
- a replacement move that adds a lower row to a higher row

are upper triangular.

Proof. The elementary matrix corresponding to scaling, say, the i^{th} row by k is the matrix with ones on the diagonal, except in the i^{th} row, which instead has a k, and zeros everywhere else. In particuar, then, the (i, j)-th entry is zero when $i > j$.

Next consider a replacement that adds the i_1 row to the i_2 row, where $i_1 < i_2$. This matrix has nonzero entries only along the diagonal and in the (i_1, i_2) position, which is above the diagonal. Therefore, all entries below the diagonal are zero. □

(b) Prove that the two kinds of row operations listed above are sufficient to row-reduce A to the identity matrix. In particular, the matrix A is invertible.

Proof. Since A is an upper triangular matrix with nonzero diagonal entries, it is already in echelon form. Therefore, we only need to perform Step 5 of the Row Reduction Algorithm on A (see page 19 of the book). This final step of the algorithm only involves row operations of the type listed above.

Moreover, every diagonal entry of A is a pivot, so we know that every diagonal entry of the reduced echelon form of A is also a pivot. The only reduced echelon form $n \times n$ matrix where every diagonal entry is a pivot is the $n \times n$ identity matrix I_n. □

(c) Prove that the product of two upper triangular matrices is upper triangular.

Proof. Suppose that U and V are two upper triangular $n \times n$ matrices. By the row-column rule for matrix multiplication we know that the (i, j)-th entry of the product UV is

$$u_{i1}v_{1j} + u_{i2}v_{2j} + \cdots + u_{in}v_{nj}.$$

We need to show that if $i > j$ then this expression evaluates to 0. In fact, we will show that every term $u_{ik}v_{kj}$ of this expression evaluates to 0.

To prove this, we consider two cases:

- If $i > k$ then $u_{ik} = 0$ since U is upper triangular. Hence $u_{ik}v_{kj} = 0$.
- If $k > j$ then $v_{kj} = 0$ since V is upper triangular. Hence $u_{ik}v_{kj} = 0$.

Since $i > j$, for every k we either have $i > k$ or $k > j$ (possibly both) so these two cases cover all possibilities for k. □

(d) Use these pieces to prove that the inverse of A is upper triangular. (You can get credit for this part even if you didn't do one of the others! Just show that you can put the pieces together to get the desired answer.)

Proof. Since A is upper triangular, we know from part (b) that there is a sequence of row operations of the type described in part (a) that transforms A into the $n \times n$ identity matrix. Therefore there is a sequence of upper triangular elementary matrices $E_1, E_2, \ldots, E_{p-1}, E_p$ such that

$$E_p E_{p-1} \cdots E_2 E_1 A = I_n.$$

As we saw in Theorem 7 of §2.2, we then have

$$A^{-1} = E_p E_{p-1} \cdots E_2 E_1.$$

The right hand side of this last equality is a product of upper triangular matrices. By part (c), the result of this product is also an upper triangular matrix. This shows that A^{-1} is an upper triangular matrix, as claimed. □