Problems Homework Set 8

Math 217 — Winter 2011

Due March 9

Problem 8.1. Let \( B = \{b_1, b_2, \ldots, b_n\} \) and \( C = \{c_1, c_2, \ldots, c_n\} \) be two bases of a vector space \( V \). Prove that the coordinate vectors \( \{[b_1]_C, [b_2]_C, \ldots, [b_n]_C\} \) form a basis of \( \mathbb{R}^n \).

Proof. We need to prove that the vectors \( \{[b_1]_C, [b_2]_C, \ldots, [b_n]_C\} \) are linearly independent and that they span \( \mathbb{R}^n \).

Suppose that

\[
x_1[b_1]_C + x_2[b_2]_C + \cdots + x_n[b_n]_C = 0.
\]

By the linearity of the coordinate transformation \( [\cdot]_C \), this is the same as

\[
[x_1 b_1 + x_2 b_2 + \cdots + x_n b_n]_C = 0.
\]

Further, because the coordinate transformation is one-to-one, this implies that

\[
x_1 b_1 + x_2 b_2 + \cdots + x_n b_n = 0.
\]

Since the vectors \( b_1, b_2, \ldots, b_n \) form a basis of \( V \) and are thus linearly independent, this is possible only if \( x_1 = x_2 = \cdots = x_n = 0 \). Therefore, \( \{[b_1]_C, [b_2]_C, \ldots, [b_n]_C\} \) is also linearly independent.

To show that the vectors \( \{[b_1]_C, [b_2]_C, \ldots, [b_n]_C\} \) span \( \mathbb{R}^n \), pick an arbitrary \( w \in \mathbb{R}^n \). Since the coordinate transformation \( [\cdot]_C : V \to \mathbb{R}^n \) is onto, there exists a vector \( v \in V \) such that

\[
[v]_C = w.
\]

\( B = \{b_1, b_2, \ldots, b_n\} \) is a basis of \( V \), and so these vectors span \( V \). Thus, there exist scalars \( x_1, x_2, \ldots, x_n \) such that

\[
v = x_1 b_1 + x_2 b_2 + \cdots + x_n b_n.
\]

Applying the coordinate transformation \( [\cdot]_C \) to both sides and using its linearity we obtain

\[
w = [v]_C = [x_1 b_1 + x_2 b_2 + \cdots + x_n b_n]_C
\]

\[
= x_1 [b_1]_C + x_2 [b_2]_C + \cdots + x_n [b_n]_C.
\]

This shows that every vector \( w \in \mathbb{R}^n \) can be written as a linear combination of the vectors \( \{[b_1]_C, [b_2]_C, \ldots, [b_n]_C\} \), which means these vectors span \( \mathbb{R}^n \). \( \square \)

Problem 8.2. Let \( U, V, W \) be three vector spaces and suppose that \( T : U \to V \) and \( S : V \to W \) are linear isomorphisms (i.e. \( T \) and \( S \) are one-to-one and onto linear transformations). Prove that their composition \( S \circ T \) is also a linear isomorphism. (Recall that the composition \( S \circ T \) is the function from \( U \) to \( W \) defined by \( (S \circ T)(x) = S(T(x)) \) for all \( x \in U \). Don’t forget to show that \( S \circ T \) is a linear transformation!)
Proof. We need to show that the function $S \circ T$ is linear, one-to-one, and onto. We first show that $S \circ T$ is linear. Consider

$$(S \circ T)(x_1 + x_2) = S(T(x_1 + x_2))$$

$$= S(T(x_1) + T(x_2))$$

$$= S(T(x_1)) + S(T(x_2))$$

$$= (S \circ T)(x_1) + (S \circ T)(x_2),$$

where we have used in each middle step the fact that both $S$ and $T$ are linear.

To show that a linear transformation $S \circ T$ is one-to-one, it would suffice to show that the only vector $x \in U$ satisfying $(S \circ T)(x) = 0$ is $x = 0$ itself. As

$$S(T(x)) = 0$$

and $S$ is one-to-one, we deduce that $T(x) = 0$. Now, because $T$ is one-to-one, this means that $x = 0$, as required.

To show that $S \circ T$ is onto, we need to prove that the equation $(S \circ T)(x) = S(T(x)) = y$ has at least one solution in $x \in U$ for any choice of $y \in W$. Since $S$ is onto, there is at least one $z \in V$ such that

$$S(z) = y.$$ Since $T$ is onto, there is at least one $x \in U$ such that

$$T(x) = z.$$ This particular $x$ will satisfy $(S \circ T)(x) = S(T(x)) = S(z) = y$, as required.

Problem 8.3. Let $V$ be a subspace of $\mathbb{R}^n$ with dimension $n - 1$ and let $x$ be a vector in $\mathbb{R}^n$ which is not in $V$.

(a) Show that there is a basis $\mathcal{B} = \{b_1, b_2, \ldots, b_n\}$ for $\mathbb{R}^n$ such that $\{b_1, \ldots, b_{n-1}\}$ is a basis for $V$ and $b_n = x$.

Proof. Let $\{b_1, \ldots, b_{n-1}\}$ be any basis for $V$. Since $x \notin V = \text{Span}\{b_1, \ldots, b_{n-1}\}$, we know from Theorem 4 in Section 4.3 that the set $\{b_1, \ldots, b_{n-1}, x\}$ is a linearly independent subset of $\mathbb{R}^n$. Since this is a set of size exactly $n$, it follows from Theorem 12 in Section 4.5 that $\{b_1, \ldots, b_{n-1}, x\}$ is a basis for $\mathbb{R}^n$.

(b) Use part (a) to show that there is a linear transformation $T : \mathbb{R}^n \to \mathbb{R}$ such that $T(x) = 1$ and the kernel of $T$ is $V$. 

Proof. Let $B = \{b_1, \ldots, b_n\}$ be the basis for $\mathbb{R}^n$ that we obtained in part (a). Let $T : \mathbb{R}^n \to \mathbb{R}$ be the linear transformation that simply returns the last $B$-coordinate of a vector in $\mathbb{R}^n$. In other words, $T(u) = c_n$ where

$$[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}.$$ 

We first check that this is a linear transformation. This follows from the linearity of the coordinate transformation $v \mapsto [v]_B$. Given vectors $u, v \in \mathbb{R}^n$ with

$$[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix} \quad \text{and} \quad [v]_B = \begin{bmatrix} d_1 \\ \vdots \\ d_n \end{bmatrix},$$

we have

$$[u + v]_B = [u]_B + [v]_B = \begin{bmatrix} c_1 + d_1 \\ \vdots \\ c_n + d_n \end{bmatrix}.$$ 

Therefore, $T(u + v) = c_n + d_n = T(u) + T(v)$. Given a vector $u \in \mathbb{R}^n$ with

$$[u]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_n \end{bmatrix}$$

and a scalar $r \in \mathbb{R}$, we have

$$[ru]_B = r[u]_B = \begin{bmatrix} rc_1 \\ \vdots \\ rc_n \end{bmatrix}.$$ 

Therefore, $T(ru) = rc_n = rT(u)$.

Now, we show that $T(x) = 1$. To see this, simply note that

$$[x]_B = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

since

$$x = b_n = 0b_1 + \cdots + 0b_{n-1} + 1b_n.$$ 

Clearly, we then have $T(x) = U([x]_B) = 1$.

Finally, we show that $T(v) = 0$ for every vector $v \in V$. Since $\{b_1, \ldots, b_{n-1}\}$ is a basis for $V$, for every vector $v \in V$ we have

$$[v]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_{n-1} \\ 0 \end{bmatrix}$$
for some scalars $c_1, \ldots, c_{n-1} \in \mathbb{R}$. In other words,

$$v = c_1 b_1 + \cdots + c_{n-1} b_{n-1} + 0b_n.$$ 

Therefore, $T(v) = 0$ for every $v \in V$. \hfill \Box