Proofs Homework Set 11

MATH 217 — WINTER 2011

Due March 30

Problem 11.1. Let \( A \) be an \( n \times n \) real symmetric matrix; i.e. all entries of \( A \) are real numbers and \( A^T = A \). Let \( v \in \mathbb{C}^n \) be an eigenvector of \( A \) corresponding to the eigenvalue \( \lambda \in \mathbb{C} \), and write \( \bar{v} = \begin{bmatrix} \bar{v}_1 \\ \vdots \\ \bar{v}_n \end{bmatrix} \).

(Recall that the complex conjugate of a complex number \( z = a + bi \), \( a, b \in \mathbb{R} \), is the complex number \( \bar{z} = a - bi \). See Appendix B of the book for properties of the complex conjugate.)

(a) Show that \( A \bar{v} = \bar{\lambda} \bar{v} \) (so \( \bar{v} \) is an eigenvector of \( A \) with eigenvalue \( \bar{\lambda} \)).

Proof. Taking the complex conjugate of \( A v = \lambda v \), we obtain \( A \bar{v} = \bar{\lambda} \bar{v} \). To see this, we repeatedly use the fact that \( w + \bar{z} = \bar{w} + z \) and \( wz = \bar{w} \bar{z} \) holds for all \( w, z \in \mathbb{C} \). For the right hand side, we have

\[
\bar{\lambda} \bar{v} = \begin{bmatrix} \bar{\lambda} \bar{v}_1 \\ \vdots \\ \bar{\lambda} \bar{v}_n \end{bmatrix} = \bar{\lambda} \bar{v}.
\]

For the left hand side, we have

\[
A \bar{v} = \begin{bmatrix} a_{11} \bar{v}_1 + \cdots + a_{1n} \bar{v}_n \\ \vdots \\ a_{n1} \bar{v}_1 + \cdots + a_{nn} \bar{v}_n \end{bmatrix} = \begin{bmatrix} a_{11} \bar{v}_1 + \cdots + a_{1n} \bar{v}_n \\ \vdots \\ a_{n1} \bar{v}_1 + \cdots + a_{nn} \bar{v}_n \end{bmatrix} = \lambda \bar{v},
\]

where we used the fact that all entries of \( A \) are real (so \( \bar{a}_{ij} = a_{ij} \)).

(b) Show that \( \bar{v}^T A v = \bar{\lambda} \bar{v}^T v \) and that \( \bar{v}^T A v = \lambda \bar{v}^T v \).

Proof. Taking the transpose of the equation \( A \bar{v} = \bar{\lambda} \bar{v} \) from (a), we obtain

\[
\bar{v}^T A = \bar{\lambda} \bar{v}^T.
\]

Multiplying on the right by \( v \) gives

\[
\bar{v}^T A v = \bar{\lambda} \bar{v}^T v.
\]

Multiplying the equation \( A v = \lambda v \) on the left by \( \bar{v}^T \) gives

\[
\bar{v}^T A v = \lambda \bar{v}^T v.
\]
(c) Show that $v^T v = \bar{v}_1 v_1 + \cdots + \bar{v}_n v_n$ is a positive real number.

**Proof.** Let us write $v_k = a_k + ib_k$ where $a_k, b_k \in \mathbb{R}$. Then

$$\bar{v}_1 v_1 + \cdots + \bar{v}_n v_n = (a_1 - ib_1)(a_1 + ib_1) + \cdots + (a_n - ib_n)(a_n + ib_n)$$

$$= (a_1^2 + b_1^2) + \cdots + (a_n^2 + b_n^2).$$

Since a sum of squares is nonnegative, we know that the result is a nonnegative real number. To see that the sum is nonzero, we use the fact that $v \neq 0$ (since $v$ is an eigenvector). It follows that $v_k = a_k + ib_k \neq 0$ for some $k$, and hence the summand $a_k^2 + b_k^2$ cannot be zero either. \(\square\)

(d) Conclude that $\lambda = \bar{\lambda}$ and hence $\lambda \in \mathbb{R}$.

**Proof.** From part (b), we conclude that $\bar{\lambda} v^T v = \lambda \bar{v}^T v$. Since $\bar{v}^T v$ is a nonzero number by part (c), we can divide both sides of this equation by this number to obtain that $\bar{\lambda} = \lambda$, which is only possible when $\lambda \in \mathbb{R}$. \(\square\)

Therefore, the eigenvalues of a real symmetric matrix are always real numbers.

**Problem 11.2.** For a polynomial $p(x)$ and an $n \times n$ matrix $A$, let $p(A)$ denote the matrix obtained by “plugging in” $A$ for $x$. For example, if $p(x) = x^3 - 2x^2 + 3$, then $p(A) = A^3 - 2A^2 + 3I$.

(a) Show that if $\lambda$ is an eigenvalue of an $n \times n$ matrix $A$, prove that $p(\lambda)$ is an eigenvalue of $p(A)$.

**Proof.** For conciseness, let us write

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.$$

Let $v$ be an eigenvector of $A$ with eigenvalue $\lambda$. Consider $p(A)v$. Since $A^k v = \lambda^k v$ for every $k$, we see that

$$p(A)v = a_m A^m v + a_{m-1} A^{m-1} v + \cdots + a_1 A v + a_0 I v$$

$$= a_m \lambda^m v + a_{m-1} \lambda^{m-1} v + \cdots + a_1 \lambda v + a_0 v$$

$$= (a_m \lambda^m + a_{m-1} \lambda^{m-1} + \cdots + a_1 \lambda + a_0) v$$

Therefore, $p(A)v = p(\lambda)v$ which shows that $p(\lambda)$ is an eigenvalue of the matrix $p(A)$. \(\square\)

(b) Show that if $A$ is similar to $B$, then $p(A)$ is similar to $p(B)$.

**Proof.** Suppose that $A = P^{-1}BP$, i.e. that $A$ is similar to $B$ via the invertible matrix $P$. As above, let us write

$$p(x) = a_m x^m + a_{m-1} x^{m-1} + \cdots + a_1 x + a_0.$$

Since $A^n = P^{-1}B^n P$, we see that

$$p(A) = a_m A^m + a_{m-1} A^{m-1} + \cdots + a_1 A + a_0 I$$

$$= a_m P^{-1}B^n P + a_{m-1} P^{-1}B^{m-1} P + \cdots + a_1 P^{-1}AP + a_0 P^{-1}P.$$
Factoring $P^{-1}$ on the left and factoring $P$ on the right, we obtain that

$$p(A) = a_m P^{-1} B^m P + a_{m-1} P^{-1} B^{m-1} P + \cdots + a_1 P^{-1} A P + a_0 P^{-1} P$$

$$= P^{-1} (a_m B^m + a_{m-1} B^{m-1} + \cdots + a_1 B + a_0 I) P = P^{-1} p(B) P.$$  

Therefore, $p(A)$ is similar to $p(B).$  

(c) Show that if $A$ is diagonalizable and $p(\lambda)$ is the characteristic polynomial of $A$, then $p(A)$ is the zero matrix.

Proof. The hypotheses say that $A = P^{-1} D P$ where $P$ is some invertible matrix and $D$ is the diagonal matrix

$$D = \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix}$$

with $\lambda_1, \lambda_2, \ldots, \lambda_n$ being the eigenvalues of $A$. Note that these scalars $\lambda_1, \lambda_2, \ldots, \lambda_n$ are precisely the roots of the characteristic polynomial of $A$, namely

$$p(\lambda) = \det(A - \lambda I) = a_n \lambda^n + a_{n-1} \lambda^{n-1} + \cdots + a_1 \lambda + a_0.$$

By part (b), we see that $p(A)$ is similar to the matrix $p(D)$. Now we can readily compute $p(D)$ as follows:

$$p(D) = a_n D^n + \cdots + a_1 D + a_0 I$$

$$= a_n \begin{bmatrix}
\lambda_1^n & 0 & \cdots & 0 \\
0 & \lambda_2^n & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n^n
\end{bmatrix} + \cdots + a_1 \begin{bmatrix}
\lambda_1 & 0 & \cdots & 0 \\
0 & \lambda_2 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & \lambda_n
\end{bmatrix} + a_0 \begin{bmatrix}
1 & 0 & \cdots & 0 \\
0 & 1 & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & 1
\end{bmatrix}$$

$$= \begin{bmatrix}
p(\lambda_1) & 0 & \cdots & 0 \\
0 & p(\lambda_2) & \cdots & 0 \\
\vdots & \ddots & \ddots & \vdots \\
0 & 0 & \cdots & p(\lambda_n)
\end{bmatrix}.$$  

Since $p(\lambda_i) = 0$ for $i = 1, 2, \ldots, n$, we see that $p(D)$ is simply the zero matrix. Therefore, $p(A) = P^{-1} p(D) P$ is the zero matrix too.  