Proofs Homework Set 13

Math 217 — Winter 2011

Due April 13

Problem 13.1. Let $B = \{b_1, \ldots, b_n\}$ be a basis for $\mathbb{R}^n$. Recall from the previous assignment that we defined the inner product $\langle u, v \rangle_B = [u]_B \cdot [v]_B$.

(a) Find a matrix $M$ such that $\langle u, v \rangle_B = u^T M v$.

(b) Show that if $B = \{b_1, \ldots, b_n\}$ is an orthonormal basis, then $\langle u, v \rangle_B = u \cdot v$ for all $u, v \in \mathbb{R}^n$.

(c) Find an example to show that part (b) is not necessarily true if $B$ is not an orthonormal basis.

(d) Suppose now that $B = \{b_1, \ldots, b_n\}$ is an orthonormal basis for $\mathbb{R}^n$ and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear transformation. Prove that the $i, j$ entry of $[T]_B$ is $\langle T(b_j), b_i \rangle_B$.

Problem 13.2. Let $W$ be a subspace of $\mathbb{R}^n$, and let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation given by $T(x) = \text{proj}_W(x)$.

(a) Show that for every $x \in \mathbb{R}^n$, $||T(x)|| \leq ||x||$.

(b) Show that for every $x \in \mathbb{R}^n$, $x \cdot T(x) \geq 0$.

(c) Define $S : \mathbb{R}^n \to \mathbb{R}^n$ by $S(x) = x - T(x)$. Show that this is the orthogonal projection onto $W^\perp$.

(d) Show that $||x||^2 = ||T(x)||^2 + ||S(x)||^2$. 