Winter 2014 Math 566 Problem Set 2
Due Friday January 25

1. Let $s_i \in S_n$ for $i \in \{1, 2, \ldots, n-1\}$ be the simple transposition which swaps $i$ and $i+1$ and leaves all other numbers fixed. Recall that $S_n$ is a group with respect to composition of bijections. Thus $s_i w$ is obtained from $w$ by swapping the values $i$ and $i+1$ in the one-line notation for $w$. For example, $s_1 s_2 = 231 \in S_3$. Also if $w = 72148635$ then $s_5 w = 72148536$.

(a) Prove that $S_n$ is generated as a group by $s_1, s_2, \ldots, s_{n-1}$.

(b) Prove that the elements $s_i$ satisfy the relations

\[ s_i^2 = 1 \]
\[ s_i s_j = s_j s_i \quad \text{if } |i-j| > 1 \]
\[ s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1} \quad \text{for } i = 1, 2, \ldots, n-2. \]

(c) (Bonus:) Prove that these are all the relations. That is, $S_n$ is isomorphic to the abstract group with generators $s_1, s_2, \ldots, s_{n-1}$ and these relations.

(d) Define the length $\ell(w) \in \mathbb{Z}_{\geq 0}$ of $w \in S_n$ to be the smallest integer $\ell$ such that there exists an expression

\[ w = s_{i_1} s_{i_2} \cdots s_{i_{\ell}} \]

for $w$ as a product of $\ell$ simple transpositions. (The length of the identity permutation is defined to be 0.) For example, $321 = s_1 s_2 s_1 = s_2 s_1 s_2$ has length 3.

Show that $\ell(w) = \text{inv}(w)$.

2. A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of $n$ is a composition of $n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0$. For example, the partitions of 5 are

\[(5), (4, 1), (3, 2), (3, 1, 1), (2, 2, 1), (2, 1, 1, 1), (1, 1, 1, 1, 1). \]

Let $p(n)$ denote the number of partitions of $n$.

(a) Argue that we have

\[ \sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} \frac{1}{1 - x^i}. \]

(b) Prove that the number of partitions of $n$ into odd parts is equal to the number of partitions of $n$ into distinct parts.

3. A permutation $w \in S_n$ is even if $(-1)^{\text{inv}(w)} = 1$, and is odd otherwise. Show that the total number of cycles of all even permutations in $S_n$ and the total number of cycles of all odd permutations of $S_n$ differ by $(-1)^n (n-2)!$. 

4. In class it was mentioned that the Eulerian number \( A(n, k) \) has the following interpretation:

\[
A(n, k) = \text{Vol}(\{ x_1, x_2, \ldots, x_n \} \in [0,1]^n \cap \{ k - 1 \leq x_1 + x_2 + \cdots + x_n \leq k \})
\]

(1)

as the normalized volume of a slice of the hypercube in \( n \) dimensions. (Normalized means that the slice for \( k = 1 \), which is a simplex, is defined to have volume 1.)

We shall prove the above statement in this problem.

(a) For each \( i \), define \( y_i = x_1 + x_2 + \cdots + x_i \), and define \( z_i = y_i - \lfloor y_i \rfloor \in [0,1) \). Check that for nearly all points \( (x_1, x_2, \ldots, x_n) \in [0,1]^n \), the vector \( (z_1, z_2, \ldots, z_n) \) determines \( (x_1, x_2, \ldots, x_n) \). Conversely, nearly all \( (z_1, z_2, \ldots, z_n) \in [0,1)^n \) comes from a unique point \( (x_1, x_2, \ldots, x_n) \in [0,1]^n \). Here “nearly all” means except for a union of lower-dimensional pieces.

(b) Say that \( z = (z_1, z_2, \ldots, z_n) \) has descent set \( D \) if \( z_i > z_{i+1} \) exactly when \( i \in D \).

Prove that if \( (x_1, x_2, \ldots, x_n) \) satisfies \( k - 1 \leq x_1 + x_2 + \cdots + x_n \leq k \) if and only if the descent set of \( (z_1, z_2, \ldots, z_n) \) has size \( k - 1 \).

(c) Prove that the map \( \phi : (x_1, x_2, \ldots, x_n) \mapsto (z_1, z_2, \ldots, z_n) \) is a measure-preserving map from \( [0,1]^n \) to \( [0,1)^n \). (Hint: the domain can be split up into pieces where the map \( \phi \) is linear. To check that a linear map is measure-preserving one just needs to show that the determinant is \( \pm 1 \). Finally, note that we can always ignore some lower-dimensional pieces since they have measure zero.)

(d) Conclude the equality (1), by cutting up the \( z \)-space appropriately.