Problem Set 2
Due on Wednesday Sept 25

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Problem 1. Prove that the following three sets are identical:

1. $GL_n(\mathbb{R})_{>0}$
2. $GL_n(\mathbb{R})_{>0} \cap B w_0 B \cap B^{-1} w_0 B^{-1}$ where $w_0$ denotes the longest element $n(n-1) \cdots 21$ of $S_n$, and $B$ denotes the upper triangular matrices in $GL_n$.
3. $U_{>0} T_{>0} U_{>0}$ where $U$ are the lower triangular matrices with 1-s on the diagonal, and $T_{>0}$ is the set of diagonal matrices with positive diagonal entries.

Problem 2. Show that if $f(t)$ is a totally positive function, then so is $(f(-t))^{-1}$.

Problem 3. A polynomial function $p(x_{ij})$ in variables $x_{ij}$ is called totally nonnegative if $p(X) \geq 0$ for any TNN matrix $X$.

1. Let $w, v \in S_n$. Prove that $x_{1,w} x_{2,w} \cdots x_{n,w(n)} = x_{1,v(1)} x_{2,v(1)} \cdots x_{n,v(n)}$ is TNN if $w \leq u$ in Bruhat order on $S_n$. (Hint: if $w < u$ in Bruhat order, then there is a chain $w = w_0 < w_1 < w_2 < \cdots < w_r = u$ where $w_i = w_{i+1}(ij)$; that is, successive permutations in the chain differ by a transposition.)

2. (This is not too hard.) Prove the converse of the previous statement.

3. (This is not too hard.) The set of all totally nonnegative polynomials forms a cone: it is closed under addition, and multiplication by $\mathbb{R}_{>0}$. Compute this cone for $2 \times 2$ and $3 \times 3$ matrices.

Problem 4. A complete matching (just “matching” in this problem) on $[2n]$ is a set of edges in the complete graph $K_{2n}$ with vertex set $[2n]$ which uses each vertex exactly once.

1. Prove that the number of matchings on $[2n]$ is $(2^{2n}-1)(2^{2n-3})\cdots 3 \cdot 1$.
2. Let $\pi$ be a matching. The crossing number $c(\pi)$ of $\pi$ is the number of (pairwise) intersections of edges when $\pi$ is drawn in a disk, with the vertices arranged in circular order on the boundary of the disk. For a skew-symmetric matrix $A$, define the pfaffian

$$\text{pf}(A) = \sum_{\pi} (-1)^{c(\pi)} \prod_{(i,j) \in \pi} a_{ij}$$

where the sum is over all matchings on $[2n]$, and in the product we always take $i < j$. For example for $n = 2$, we have $\text{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23}$.

A proof of the following classical identity

$$\text{pf}(A)^2 = \det(A)$$

can be found easily online.

Let $N$ be a planar acyclic directed network as usual, with $2n$ sources and an arbitrary number of sinks. Define a skew-symmetric $2n \times 2n$ matrix $A(N)$ by setting

$$a_{ij} = \sum_{p,q} \text{wt}(p)\text{wt}(q)$$
for $i < j$, where the summation is over all pairs of noncrossing paths from sources $i$ and $j$ to any pair of sinks. Prove Stembridge’s Pfaffian-analogue of the Lindström Lemma:

$$\text{pf}(A(N)) = \sum_P \text{wt}(P)$$

where the summation is over all noncrossing families of paths $P$ using all the sources and any subset of sinks.

(3) Suppose $n = 2$ and $N$ is a planar acyclic directed network with nonnegative edge weights and 4 sources. Let $A = A(N)$. Show that $a_{13}a_{24} - a_{14}a_{23} \geq 0$. Conclude that subpfaffian positivity is not enough to guarantee that a skew-symmetric matrix $A$ is realizable by a network.

(4) (*) (This problem generalizes the previous one significantly.) Let $A = A(N)$ be a $2n \times 2n$ skew-symmetric matrix arising from a network $N$ with nonnegative edge weights.

(a) Suppose $|I| = |J|$ is even. Prove that $|A_{I,J}| \geq 0$.

(b) Suppose $|I| = |J|$ is odd. Prove that $|A_{I,J}| \geq 0$ for all networks if and only if $i_1 \leq j_1, i_2, \leq j_2, \ldots$.

(5) (*) (Open?) Find semialgebraic conditions on a $2n \times 2n$ skew-symmetric matrix that guarantee realizability by a network. For example, are the conditions of the previous problem, together with nonnegativity of subpfaffians enough to guarantee realizability?

(6) (*) (Open?) Find “generators” for the set of $2n \times 2n$ skew-symmetric matrices that are realizable as $A(N)$ by a planar network.

**Problem 5.** (*) (This problem is not hard, just optional.) Suppose $X$ is a $n \times n$ matrix. Fix $I, J \subset [n], |I| = |J| = r$ and for $i \in [n]/I, j \in [n]/J$ let

$$y_{i,j} = |X_{I\cup i, J\cup j}|.$$

We assume the following basic determinantal identity (Sylvester’s identity):

$$\det(Y) = |X_{I,J}|^{n-r-1}|X|.$$

(1) Let $X$ be a $n \times (n + 1)$ matrix. Fix integers $1 < k, \ell < n + 1$. Use Sylvester’s identity to prove

$$|X_{[n], [n+1]/l}|X_{[n]/k, [n]/l}| = |X_{[n], [n+1]/l}|X_{[n]/k, [n]/l}| + |X_{[n], [n]}|X_{[n]/k, [n+1]/(1,l)}|.$$

(Hint: apply Sylvester’s identity with $r = n - 1$ to the matrix obtained from $X$ by adding a row with entries $(0, 0, \ldots, 0, 1)$.)

(2) Use (1) to prove the following Lemma due to Fekete. Assume $X$ is an $n \times m$ matrix with $n \geq m$, such that the minors $|X_{I,[m-1]}|$ for any $I$ are positive, and all minors of size $m$ with consecutive (solid) rows are positive. Then all minors of $X$ of size $m$ are positive.

(3) Use (2) to obtain Fekete’s criterion for total positivity: if all solid minors of $X$ are positive, then $X$ is TP.