Problem Set 7
Due on November 15

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Problem 1. This problem (partly) bridges the gap between matchings and non-intersecting paths.

Let $G$ be a planar bipartite graph as considered in class. An orientation $O$ of the edges of $G$ is \textbf{perfect} if interior black vertices have outdegree 1 (and any indegree), while interior white vertices have indegree 1 (and any outdegree).

1. Prove that almost perfect matchings are naturally in bijection with perfect orientations. Under this bijection, what is the boundary subset $I(\Pi)$ sent to?
2. Let $O$ be a perfect orientation of $G$. A \textbf{flow} $F$ in $(G, O)$ is a collection of edges such that the number of incoming edges is equal to the number of outgoing edges, at any interior vertex of $G$. Prove that flows in $(G, O)$ are in bijection with perfect orientations $O'$ of $G$.
3. With a perfect orientation $O$ of $G$ fixed, express the boundary measurements $\Delta_I(N)$ in terms of flows in $(G, O)$.
4. Suppose $(G, O)$ is acyclic. How are flows in $(G, O)$ related to non-intersecting paths?

Problem 2. Show that the relation $x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}(a')x_i(b')x_{i+1}(c')$ for Chevalley generators can be deduced from the relations for bipartite graphs.

Problem 3. A \textbf{matroid} (of rank $k$ on $[n]$) is a collection $\mathcal{M}$ of $k$-element subsets of $[n]$, satisfying the \textbf{exchange axiom}: given $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.

1. Let $X$ be a $k \times n$ matrix. Prove that $\mathcal{M}_X = \{I \mid \Delta_I(X) \neq 0\}$ is a matroid.
2. Let $\mathcal{M}$ be a matroid. Show that $\mathcal{M}$ satisfies the \textbf{dual exchange axiom}: if $I, J \in \mathcal{M}$ and $j \in J$ there exists $i \in I$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.
3. Let $\mathcal{M}$ be a matroid. Show that $\mathcal{M}$ satisfies the \textbf{symmetric exchange axiom}: if $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that both $(I - \{i\} \cup \{j\})$ and $(J - \{j\} \cup \{i\})$ belong to $\mathcal{M}$.
4. (*) For a subset $I \subset [n]$, let $e_I = \sum_{i \in I} e_i$ be the 0-1 vector with 1-s in the positions specified by $I$.

The matroid polytope of $\mathcal{M}$ is the the convex hull $\text{conv}(e_I \mid I \in \mathcal{M})$. Prove that a polytope with vertices given by 0-1 vectors is a matroid polytope if and only if all edges are parallel to $e_i - e_j$ for some $i, j \in [n]$.

Problem 4. Fix $k, n$ as usual. Let $f : \mathbb{Z} \rightarrow \mathbb{Z}$ be a bounded affine permutation, as in Section 2.5 of the notes. Prove that $T_a = \{b < a \mid f(b) \geq a\}$ has the same cardinality, for any $a \in \mathbb{Z}$. 

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