Problem Set 7
Due on November 15

All non-starred problems are due on the above date. Starred problems can be handed in anytime before December 6.

Problem 1. This problem (partly) bridges the gap between matchings and non-intersecting paths.

Let G be a planar bipartite graph as considered in class. An orientation O of the edges of G is **perfect** if interior black vertices have outdegree 1 (and any indegree), while interior white vertices have indegree 1 (and any outdegree).

(1) Prove that almost perfect matchings are naturally in bijection with perfect orientations. Under this bijection, what is the boudary subset $I(\Pi)$ sent to?
(2) Let O be a perfect orientation of G. A **flow** F in (G, O) is a collection of edges such that the number of incoming edges is equal to the number of outgoing edges, at any interior vertex of G. Prove that flows in (G, O) are in bijection with perfect orientations O' of G.
(3) With a perfect orientation O of G fixed, express the boundary measurements $\Delta_I(N)$ in terms of flows in (G, O).
(4) Suppose (G, O) is acyclic. How are flows in (G, O) related to non-intersecting paths?

Problem 2. Show that the relation $x_i(a)x_{i+1}(b)x_i(c) = x_{i+1}(a')x_i(b')x_{i+1}(c')$ for Chevalley generators can be deduced from the relations for bipartite graphs.

Problem 3. A **matroid** (of rank k on $[n]$) is a collection $\mathcal{M}$ of k-element subsets of $[n]$, satisfying the **exchange axiom**: given $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that $(I - \{i\} \cup \{j\}) \in \mathcal{M}$.

(1) Let X be a $k \times n$ matrix. Prove that
$$\mathcal{M}_X = \{I \mid \Delta_I(X) \neq 0\}$$
is a matroid.
(2) Let $\mathcal{M}$ be a matroid. Show that $\mathcal{M}$ satisfies the **dual exchange axiom**: if $I, J \in \mathcal{M}$ and $j \in J$ there exists $i \in I$ such that $(I - \{i\} \cup \mathbf{j}\}) \in \mathcal{M}$.
(3) Let $\mathcal{M}$ be a matroid. Show that $\mathcal{M}$ satisfies the **symmetric exchange axiom**: if $I, J \in \mathcal{M}$ and $i \in I$, there exists $j \in J$ such that both $(I - \{i\} \cup \{j\})$ and $(J - \{j\} \cup \{i\})$ belong to $\mathcal{M}$.
(4) **(*)** For a subset $I \subset [n]$, let $e_I = \sum_{i \in I} e_i$ be the 0-1 vector with 1-s in the positions specified by I.

The matroid polytope of $\mathcal{M}$ is the the convex hull $\text{conv}(e_I \mid I \in \mathcal{M})$. Prove that a polytope with vertices given by 0-1 vectors is a matroid polytope if and only if all edges are parallel to $e_i - e_j$ for some $i, j \in [n]$.

Problem 4. Fix k, n as usual. Let $f : \mathbb{Z} \to \mathbb{Z}$ be a bounded affine permutation, as in Section 2.5 of the notes. Prove that
$$T_a = \{b < a \mid f(b) \geq a\}$$
has the same cardinality, for any $a \in \mathbb{Z}$.