A proof of Problem Set 5 Problem 4 part 4.

Lemma 1. *Suppose $a_i \in \mathbb{R}$ satisfies $\sum_i |a_i| < \infty$, and let r be an integer, and $a \in \mathbb{R}$. Then $f(t) = at^r \prod_i (1 + a_i t)$ grows slower then any exponential* $\exp(ct)$ *for* $c > 0$.

Proof. (An attempt to not use any theorems in complex analysis.) We take for granted that any polynomial grows slower than any exponential (a theorem from calculus?). So by modifying c slightly, we can always throw away finitely many factors from $f(t)$. Thus we assume that $\sum_i |a_i| < c/2$, say, and we also throw away that at^r.

It is enough to show that we have $\exp(ct) > \prod_{i=1}^n (1 + |a_i t|)$ for $t \in \mathbb{R}_{>0}$ (since by taking a limit, we will get that $\exp(ct) \geq |f(t)|$).

But $\exp(ct) > (1 + ct/k)^k$ for any k. By picking k very large, we can arrange for each $a_1, a_2, \ldots, a_n$ to satisfy $b_i c/k \leq |a_i| < (b_i + 1)c/k < 2|a_i|$ for some integer $b_i \geq 1$. But clearly $(1 + ct/k)^{b_i+1} > (1 + |a_i|t)$, and the assumption $\sum_i |a_i| < c/2$ shows that $(1 + ct/k)^k > \prod_{i=1}^n (1 + |a_i t|)$. □

After cancelling denominators in the formula for the determinant of a 2×2 matrix $A(t)$, we get (using part 3)

$$\exp(\gamma t) f(t) = \exp(\alpha t) g(t) - \exp(\beta t) h(t)$$

where f, g, h are infinite products as in the Lemma, and $\alpha, \beta \geq 0$. If $\gamma < 0$, then to avoid the degenerate case where α, β could be 0, we write

$$\exp(\gamma t/2) f(t) = \exp((\alpha - \gamma/2)t) g(t) - \exp((\beta - \gamma/2)t) h(t)$$

and apply the Lemma. The LHS is unbounded as $t \to -\infty$, but the RHS goes to 0. So we get a contradiction, and we must have $\gamma \geq 0$.

Now $\det(A(t)^{-1}) = 1/\det(A(t))$, and if $A(t)$ is 2×2, then $\det(A(t)^{-c}) = 1/\det(A(t))$ as well. Thus the same argument for $A(t)^{-c}$ gives $\gamma \leq 0$, so $\gamma = 0$.

Repeating the argument we get $\alpha = \beta = 0$.