Lemma 1. *Let ϕ be a normalized harmonic function, and suppose that there is a partition λ with at least 2 boxes such that $\phi(\lambda) = 1$. Then $\phi = \phi_h$ or ϕ_e, given by*

$$\phi_h(\mu) = \begin{cases} 1 & \mu \text{ is a single row} \\ 0 & \text{otherwise.} \end{cases}$$

$$\phi_e(\mu) = \begin{cases} 1 & \mu \text{ is a single column} \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The single rows and single columns are exactly those partitions λ such that there exists a unique (increasing, i.e. always add boxes) path from the empty partition $\emptyset$ to λ. The claim then follows from the definition of harmonic. □

Let $\psi_h : \mathrm{Sym} \to \mathbb{R}$ be the ring homomorphism given by sending $h_1, h_2, \ldots$ all to 1. By looking at the minors of the infinite Toeplitz matrix

$$\begin{pmatrix} 1 & 1 & 1 & 1 & \cdots \\ 0 & 1 & 1 & 1 & \cdots \\ 0 & 0 & 1 & 1 & \cdots \\ 0 & 0 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{pmatrix}$$

we see that

$$\psi_h(s_\lambda) = \begin{cases} 1 & \lambda \text{ is a single row} \\ 0 & \text{otherwise.} \end{cases}$$

So $\phi_h = \psi_h$ is a ring homomorphism. This ring homomorphism can also be described as

$$\phi_h(f(x_1, x_2, x_3, \ldots)) = f(1, 0, 0, 0, \ldots).$$

The corresponding character $\chi_h : S_\infty \to \mathbb{C}$ is the function that is identically equal to 1. The corresponding TP function is $f(t) = 1/(1-t)$. Exercise: chase through the definitions until this is clear.

The map ϕ_e is also a ring homomorphism: it is given by sending all the elementary symmetric functions e_i to 1. It is related to ϕ_h by the operation in Problem 2 of Problem Set 2. The corresponding character χ_e is the sign character of S_∞. The corresponding TP function is $f(t) = 1 + t$.