Cross ratios. Recall that the cross ratio is defined by
\[ [1, 2; 3, 4] := \frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}. \]

1. (a) For the 4! permutations \( w_1, w_2, w_3, w_4 \) of 1, 2, 3, 4, write down how the cross ratio \([w_1, w_2; w_3, w_4]\) is related to \([1, 2; 3, 4]\).
(b) Show that \([i, j; k, ℓ] = [i, j; k, m][i, j; m, ℓ]\).

2. Show that the equations from the previous problem cut \( \mathcal{M}_{0,n} \) out of \( \mathcal{M}_0(n) \). That is, (a) show that a point in \( \mathcal{M}_{0,n} \) is uniquely determined by its cross ratios, and (b) show that a collection of \( \binom{n}{4} \) numbers is the set of cross ratios of a point in \( \mathcal{M}_{0,n} \) if and only if it is consistent with these equations. (Note that 1(a) actually gives you \( n! \) numbers.)

Compactification. For this problem, define the compactification \( \mathcal{M}_{0,n} \) as the closure of \( \mathcal{M}_{0,n} \) inside \( \mathcal{M}_0(n) = \mathbb{P}^1(\mathbb{P}^1) \).

3. Show that every point in \( \mathcal{M}_{0,n} \) is represented by a stable rational curve with \( n \) marked points.
4. Each stable rational curve \( C \) with \( n \) marked points gives rise to a tree \( T(C) \) whose interior vertices correspond to components, and leaves correspond to marked points. Fix a such a tree \( T \). Find and prove a formula for the dimension of the stratum of \( \mathcal{M}_{0,n} \) consisting of points represented by stable curves with tree \( T(C) = T \).

Dihedral coordinates. Define the dihedral coordinates \( u_{ij} := [i, i-1; j-1, j] \) for \( (i, j) \) a diagonal of the \( n \)-gon.
5. Prove the equation
\[ u_{ij} + \prod_{(k,ℓ)\text{crosses}(i,j)} u_{kℓ} = 1. \]
6. Let \( I_n \subset \mathbb{C}[u_{ij}] \) be the ideal generated by the relations of the previous problem. Define \( \widetilde{\mathcal{M}}_{0,n} := \text{Spec}(\mathbb{C}[u_{ij}] / I_n) \). Prove that for any partition of \([n]\) into cyclic intervals \( A, B, C, D \), appearing in cyclic order, we have
\[ \prod_{a \in A, c \in C} u_{ac} + \prod_{b \in B, d \in D} u_{bd} - 1 \in I_n. \]
7. Show that on \( \mathcal{M}_{0,n} \) any three of the four cross ratios \( u_{ab}, u_{a+1,b}, u_{a,b+1}, u_{a+1,b+1} \) determines the fourth.
8. Show that \( \widetilde{\mathcal{M}}_{0,n} \) is a smooth complex algebraic variety for \( n = 4, 5 \). (Bonus: do this for general \( n \), which I don’t think is easy directly.)
9. Show that the boundary \( \widetilde{\mathcal{M}}_{0,n} \setminus \mathcal{M}_{0,n} \) is a simple normal-crossing divisor. Do this for \( n = 4, 5 \), and as a bonus in general.
10. Let \( u'_{ij} \) be the dihedral coordinates for the ordering of 2, 1, \ldots, \( n \) of the \( n \) marked points. Find formulae that express \( u_{ij} \) in terms of \( u'_{ij} \) and vice-versa.
11. Let \( \widetilde{\mathcal{M}}_{0,n}^\omega \) denote the version of \( \widetilde{\mathcal{M}}_{0,n} \) for a (dihedral) ordering \( \omega \) of 1, 2, \ldots, \( n \). Show that \( \mathcal{M}_{0,n} \) is covered by the \( \widetilde{\mathcal{M}}_{0,n}^\omega \), as \( \omega \) varies over all dihedral orderings.
Canonical form. Recall that the canonical form $\Omega = \Omega((\mathcal{M}_{0,n})_{\geq 0}) = \prod_i \text{dlog} y_i$ where $y_i$ are the coordinates of the positive parametrization, e.g. for $n = 5$:

$$
\begin{pmatrix}
1 & 1 & 1 & 1 & 0 \\
0 & 1 & 1 + y_1 & 1 + y_1 + y_1 y_2 & 1
\end{pmatrix}
$$

12. Show that the subvariety of $\widetilde{\mathcal{M}}_{0,n}$ cut out by $u_{ij} = 0$ is isomorphic to $\widetilde{\mathcal{M}}_{0,j-i+1} \times \widetilde{\mathcal{M}}_{0,n-j+i+1}$.

13. Prove that

$$
\Omega = \pm \prod_{i=3}^{n-1} \frac{\text{d}u_{1,i}}{u_{1,i}(1 - u_{1,i})}.
$$

14. Show that $\text{Res}_{u_{ij}=0} \Omega$ is the canonical form $\Omega((\mathcal{M}_{0,j-i+1})_{\geq 0}) \wedge \Omega((\mathcal{M}_{0,n-j+i+1})_{\geq 0})$ under the earlier isomorphism.

15. Show that $(\overline{\mathcal{M}}_{0,n}, (\mathcal{M}_{0,n})_{\geq 0})$ is a positive geometry. You may assume that none of the strata have singularities. The hard part is to show that $\Omega$ has no poles on $\overline{\mathcal{M}}_{0,n}$ except the ones in the previous problem.