(1) **Computational exercises.** Verify directly that the quadrilateral $P$ with the four vertices $(0,0), (2,0), (0,1), (1,2)$ has canonical form

$$\Omega = C \frac{y - 4x - 4}{xy(y - x - 1)(2x + y - 4)}dxdy$$

for some nonzero constant $C$.

(2) Compute the canonical form of the pentagon with vertices $(0,0), (1,0), (2,1), (1,2), (0,1)$ by solving the residue equations.

(3) Verify that the pizza slice bounded by the lines $\sqrt{3}y + x, \sqrt{3}y - x$ and the circle $x^2 + y^2 = 1$ is given by

$$\Omega = C \frac{1 + 2y}{(1 - x^2 - y^2)(\sqrt{3}y + x)(\sqrt{3}y - x)}dxdy$$

for some nonzero constant $C$. (Note that there’s more than one possible pizza slice that I could be talking about!)

(4) **Duality.** For a cone $C \subset \mathbb{R}^{n+1}$ we define the dual cone by

$$C^\vee = \{ \mathbf{x} \in \mathbb{R}^{n+1} | \mathbf{x} \cdot \mathbf{y} \geq 0 \text{ for all } \mathbf{y} \in C \}. $$

Show that if $C$ is a pointed, full-dimensional, polyhedral cone, so is $C^\vee$. If $P \subset \mathbb{P}^n$ is a projective polytope, we define the dual $P^\vee$ as follows: take a cone $C \subset \mathbb{R}^{n+1}$ such that the image of $C$ in $\mathbb{P}^n$ is $P$. Define $P^\vee$ to be the image of $C^\vee$ in $\mathbb{P}^n$.

(5) Recall that for $P \subset \mathbb{R}^n$ containing 0 in its interior, we define the polar polytope

$$P^\vee = \{ \mathbf{x} \in \mathbb{R}^n | \mathbf{x} \cdot \mathbf{y} \geq -1 \text{ for all } \mathbf{y} \in P \}. $$

Show that duality of cones is compatible with polar polytopes: let $C(P) = \{ (t,t\mathbf{x}) | t \in \mathbb{R}_{\geq 0}, \mathbf{x} \in P \} \subset \mathbb{R}^{n+1}$. Find the relation between $C(P)^\vee$ and $P^\vee$.

(6) Let $C \subset \mathbb{R}^{d+1}$ be a pointed cone. An orientation of $C$ gives rise to an orientation of the projective polytope $P(C) \subset \mathbb{P}^d$. We fix a “positive” orientation of $\mathbb{R}^{d+1}$. An orientation of $C$ is positive if it agrees with that of $\mathbb{R}^{d+1}$. If $C$ is positively oriented, we declare that $P(C)$ is as well.

Verify the following statement: if $g \in \text{GL}(d+1)$ and $C$ is a positively oriented cone, then $g \cdot C$ is positively (resp. negatively) oriented if $\det(g) > 0$ (resp. $\det(g) < 0$).

(7) Let $P$ be a projective polytope. Prove that a signed triangulation of $P$ gives rise to a signed triangulation of $P^\vee$.

$$P = \sum_i \pm P_i \implies P^\vee = \sum_i \pm P_i^\vee$$

All projective polytopes here are oriented, and $-P$ denotes the same polytope with the opposite orientation. (This is a bit of an abuse of notation, since $-P$ also has other meanings.)
(8) **Forms on projective space.** Let \( \omega \) be a rational \( m \)-form on \( \mathbb{P}^m \). Let \( \pi : \mathbb{C}^{m+1} - \{0\} \to \mathbb{P}^m \) denote the projection map. Then the pullback \( \pi^*(\omega) \) is a rational \( m \)-form on \( \mathbb{C}^{m+1} - \{0\} \) that extends to a rational \( m \)-form on \( \mathbb{C}^{m+1} \). Thus,

\[
\pi^*(\omega) = \sum_{i=0}^{m} \frac{P_i(x)}{Q_i(x)} dx_0 \wedge \cdots \wedge \tilde{d}x_i \wedge \cdots \wedge dx_m
\]

for rational functions \( P_i/Q_i \). Show that

(a) \( P_i/Q_i \) has degree \(-m\), and

(b) \( \pi^*(\omega) \) contracts to 0 with the radial vector field \( \sum_i x_i \partial_i \) and that pullbacks of \( m \)-forms on \( \mathbb{P}^m \) are characterized by these properties.

(9) (Continuation of previous problem.) Show that conditions (a) and (b) are equivalent to

\[
\pi^*(\omega) = \frac{P(x)}{Q(x)} \sum_{i=0}^{m} (-1)^i x_i dx_0 \wedge \cdots \wedge \tilde{d}x_i \wedge \cdots \wedge dx_m = \frac{P(x)}{Q(x)} \frac{1}{m!} (xd^m x)
\]

for \( P/Q \) of degree \(-m - 1\). For example, if \( m = 1 \), then

\[
\pi^*(\omega) = \frac{P(x)}{Q(x)} (x_0 dx_1 - x_1 dx_0) = \frac{P(x)}{Q(x)} \det \begin{pmatrix} x_0 & x_1 \\ dx_0 & dx_1 \end{pmatrix}.
\]

(10) Let \( P \subset \mathbb{P}^m \) be a projective polytope. Prove the formula

\[
\Omega(P) = \frac{1}{m!} \left( \int_{P^r \subset \mathbb{P}^m} \frac{\langle y d^m y \rangle}{(x, y)^{m+1}} \right) (xd^m x)
\]

Here, the coordinate on \( P \) is denoted \( x \), and the coordinate on \( P^r \) is denoted \( y \).

(11) **Moments of polytopes.** The moment\( s \) of a polytope \( P \subset \mathbb{R}^m \) are defined by

\[
m_I(P) := \int_P y_1^{i_1} \cdots y_m^{i_m} dy_1 \cdots dy_m.
\]

Relate the canonical form of \( P \) and the moments of \( P^r \) as follows. Place \( P^r \) in the chart \((-1, y')\) and \( P \) in the chart \((1, x')\). Then

\[
\Omega(P) = \frac{1}{m!} \left( \int_{P^r \subset \mathbb{P}^m} \frac{\langle y d^m y \rangle}{(x, y)^{m+1}} \right) = \frac{1}{m!} \left( \int_{P' \subset \mathbb{R}^m} \frac{dy_1 dy_2 \cdots dy_m}{(-1 + x' y')^{m+1}} \right) = \sum_{I \in \mathbb{Z}^m_{\geq 0}} a_I m_I(P^r) x_1^{i_1} \cdots x_m^{i_m}
\]

for some constants \( a_I \) not depending on \( P, P^r \).

(12) **Barycentric coordinates.** If \( v_1, \ldots, v_{n+1} \) are vertices of a simplex in \( \mathbb{R}^n \), every point in the interior of the simplex can be written uniquely as

\[
x = b_1 v_1 + \cdots + b_{n+1} v_{n+1}, \quad b_1 + \cdots + b_{n+1} = 1
\]

where \( b_1, b_2, \ldots, b_{n+1} \) are the barycentric coordinates of \( x \). But what if the simplex is replaced by a (full-dimensional) polytope \( P \subset \mathbb{R}^n \) with vertices \( v_1, \ldots, v_r \)? Show that volumes of the natural decomposition of \( (P - x)^r \) indexed by vertices gives barycentric coordinates \( b_i(x) \) satisfying (for \( x \in P \)):

(a) \( b_i(x) \geq 0 \)

(b) \( b_1(x) + \cdots + b_r(x) = 1 \)

(c) \( b_1(x)v_1 + \cdots + b_r(x)v_r = x \).