Polytope canonical forms

Let $P \subset \mathbb{P}^d$ be a (oriented) projective polytope. Then $P$ has a canonical form $\Omega(P)$, which is a rational $d$-form on $\mathbb{P}^d$.

1. (Residue definition) $\Omega(P)$ has poles only along facet hyperplanes, these poles are simple, and we have
   \[ \text{Res}_H \Omega(P) = \Omega(F) \]
   where $H$ is a facet hyperplane and $F = H \cap P$ is a facet.

2. (Subdivisions or triangulations) Suppose $P$ is subdivided into polytopes $T_1, T_2, \ldots, T_r$. Then
   \[ \Omega(P) = \sum_{i=1}^r \Omega(T_i). \]

3. (Dual volume) Suppose $P \subset \mathbb{R}^d \subset \mathbb{P}^d$. Then
   \[ \Omega(P)(x) = \text{Vol}((P - x)^\vee) d^d x. \]
   where $x \in \mathbb{R}^d$. Here, Vol denotes normalized volume and $Q^\vee$ is the dual (or polar) polytope of $Q$.

4. (Laplace transform) Let $C \subset \mathbb{R}^{d+1}$ denote the cone over $P$. Then
   \[ \Omega(P) = \frac{1}{d!} \left( \int_{C^\vee} e^{-x^T y} dy \right) \langle x^d d^d x \rangle \]
   where $C^\vee \subset \mathbb{R}^{d+1}$ is the dual cone, and here $x, y$ are vectors in $\mathbb{R}^{d+1}$.

5. (Numerator is adjoint) Suppose $P \subset \mathbb{R}^d \subset \mathbb{P}^d$. Then
   \[ \Omega(P) = c \cdot \text{adjoint hypersurface} \frac{d^d x}{\prod_{\text{facets } H} H} \]
   for some nonzero constant $c$.

6. (Pushforward) Suppose $P$ has (projective) vertices $w_1, w_2, \ldots, w_m$, and let \{$(1, v_1), (1, v_2), \ldots, (1, v_m)$\} be vectors in $\mathbb{R}^{d+1}$ with the same oriented matroid as \{$(w_1, w_2, \ldots, w_m)$\} i.e. $\text{sign det}(w_{i_1}, \ldots, w_{i_{d+1}}) = \text{sign det}((1, v_{i_1}, \ldots, (1, v_{i_{d+1}}))$ for all $\{i_1, \ldots, i_{d+1}\}$. Define the rational map $\Phi_V : (\mathbb{C}^\times)^d \to \mathbb{P}^d$ by
   \[ \Phi : (x_1, \ldots, x_d) \mapsto \sum_{i=1}^m x^{v_i} w_i. \]
   Then
   \[ \Omega(P) = \Phi_* \prod_i d\log x_i. \]