Mirror symmetry for flag varieties
via
Langlands reciprocity

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This is joint work with Nicolas Templier.
Quantum differential equations

$M$ smooth compact Fano variety over $\mathbb{C}$
Examples: $\mathbb{P}^n$, $\text{Gr}(k, n)$, $G/P$, ...
Fano index: largest integer $m$ such that $-K_X = mD$ in $\text{Pic}(M)$
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Quantum connection (Dubrovin) in anticanonical direction

Connection on the trivial $H^*(M)$-bundle over $\mathbb{C}_q^\times$:

$$\nabla = \nabla_M = d + D_q \frac{dq}{q}$$

Here, log $q$ is a coordinate on $\mathbb{C} \cdot [D] \subseteq H^2(X, \mathbb{C})$.
$\nabla$ is a connection on $\mathbb{C}_q^\times$ (regular at 0 and irregular at $\infty$).
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$M = \mathbb{P}^1$ with $\dim(H^*(\mathbb{P}^1)) = 2$

$$\left( q \frac{d}{dq} + \begin{bmatrix} 0 & q \\ 1 & 0 \end{bmatrix} \right) \begin{bmatrix} y_1(q) \\ y_2(q) \end{bmatrix} = 0$$
Landau-Ginzburg model

\[(X = \text{smooth complex variety, } f, \pi)\]

\[
\begin{array}{ccc}
X & \xrightarrow{f} & \mathbb{C} \\
\downarrow{\pi} & & \downarrow{\pi} \\
\mathbb{C}^\times_q & & \\
\end{array}
\]
Landau-Ginzburg model

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Integral functions

\[X_q := \pi^{-1}(q)\]

\[\Psi(q) := \int_{\Gamma_q \subset X_q} e^{f(x)} \omega_q\]
Landau-Ginzburg model

\[ X \overset{f}{\longrightarrow} \mathbb{C} \]
\[ \downarrow \pi \]
\[ \mathbb{C}_q^\times \]

\( (X = \text{smooth complex variety}, f, \pi) \)

**Integral functions**

\[ X_q := \pi^{-1}(q) \]
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**Landau-Ginzburg D-module**

\[ \text{Exp} := \mathbb{C}\langle x, \partial \rangle / (\partial x - x \partial - 1) = \mathbb{C}\langle x, \partial \rangle \cdot e^x \]

\[ C = C(X, f, \pi) := R\pi_! f^* \text{Exp}. \]

Object in derived category of D-modules on \( \mathbb{C}_q^\times \).
Givental’s mirror conjecture

The integral functions $\Psi(q)$ are solutions to $\nabla_M$.
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Stronger variant: $D$-module mirror conjecture

$\mathcal{C}(X, f, \pi)$ is a $D$-module and is isomorphic to $\nabla_M$ as $D$-modules on $\mathbb{C}_q^\times$. 
Main Theorem

Theorem (L.-Templier)

Mirror conjecture holds for $M = G^\vee / P^\vee$ a minuscule flag variety, and Rietsch’s LG-model $(X, f, \pi)$.

Minuscule flag varieties:

- $\mathbb{P}^n$ (classical/Givental),
- $\text{Gr}(k, n)$ (injection proved by Marsh-Rietsch),
- $\text{OG}(n, 2n + 1), \text{OG}(n, 2n)$,
- $Q^{2n}$ (injection proved by Pech-Rietsch-Williams),
- Cayley plane,
- Freudenthal variety
Rietsch’s LG-model

- $(X, f, \pi)$ is a geometric crystal of Berenstein-Kazhdan,
- $\Psi(q)$ is a geometric character,
- $C(X, f, \pi)$ is called the character $D$-module.

The fibers

$$X_q = \pi^{-1}(q) \cong G^\circ / P \subset G / P$$

are isomorphic to a log Calabi-Yau subvariety called a projected Richardson variety (Lusztig, Rietsch, Goodearl-Yakimov, Knutson-L.-Speyer,..).
Proof idea

\[ \text{B-model} \quad \begin{array}{c}
\text{character } D\text{-module of geometric crystal for } (G, P) \\
\end{array} \quad \text{Givental-Rietsch mirror conjecture} \quad \begin{array}{c}
\text{A-model} \\
\text{quantum } D\text{-module for } G^\vee / P^\vee \\
\end{array} \]

- **LT**
  - Kloosterman \( D\)-module
  - Zhu’s Theorem
- **LT**
  - Frenkel-Gross connection
  - Galois

Zhu’s theorem (based on Beilinson-Drinfeld) is an instance of the ramified geometric Langlands conjectures. Right hand side is a calculation. Depends on some computations with canonical bases, and Mihalcea’s quantum Chevalley formula. (cf. Golyshev-Manivel in simply-laced cases)
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Projective space case

\[ M = \mathbb{P}^{n-1} \quad QH^*(\mathbb{P}^{n-1}) = \mathbb{C}[x, q]/(x^n - q). \]

**quantum D-module**

\[
q \frac{d}{dq} + \begin{bmatrix}
0 & 0 & \cdots & 0 & q \\
1 & 0 & \cdots & 0 & 0 \\
0 & 1 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 1 & 0 \\
\end{bmatrix} = 0 \iff ((q \frac{d}{dq})^n - q)(\vec{y}(q)) = 0
\]

(For \( n = 1 \): Bessel equation)
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LG-model

\[
\begin{array}{ccc}
(\mathbb{C}^\times)^n & \xrightarrow{f} & \mathbb{C} \\
\downarrow \pi & & \downarrow \\
\mathbb{C}^\times_q & & x_1 x_2 \cdots x_n \\
\end{array}
\]
Kloosterman sums

Base change to $\mathbb{F}_q$

$$ (\mathbb{F}_q^\times)^n \xrightarrow{f} \mathbb{F}_q $$

$$ \xrightarrow{\pi} $$

$$ \mathbb{F}_q^\times $$

$$(x_1, x_2, \ldots, x_n) \quad \xrightarrow{\text{map}} \quad x_1 + x_2 + \cdots + x_n$$

$$ x_1 x_2 \cdots x_n $$

Kloosterman sums are analogues of the $\Psi(q)$

For $a \in \mathbb{F}_q^\times$, define

$$ K_{l,n}(a) := (-1)^{n-1} \sum_{x_1 x_2 \cdots x_n = a} \exp \left( \frac{2\pi i}{p} \text{Tr} f(x) \right) \in \mathbb{C} $$

Here, $\text{Tr} : \mathbb{F}_q \rightarrow \mathbb{F}_p$.

Weil-Deligne bound: $|K_{l,n}(a)| \leq n q^{(n-1)/2}$. 
Kloosterman sheaves

Deligne (1970s): defined Kloosterman sheaf

\[ \text{Kl}_{\mathbb{Q}_\ell}^n := R\pi_! f^* \text{AS}_\chi \]

where \( \text{AS}_\chi \) is an Artin-Schreier sheaf. For suitable \( \chi \) and \( \iota: \overline{\mathbb{Q}}_{\ell} \to \mathbb{C} \),

\[ \text{Kl}_n(a) = \iota \text{Tr}(\text{Frob}_a, (\text{Kl}_{\mathbb{Q}_\ell}^n)_a) \]
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where $\text{AS}_\chi$ is an Artin-Schreier sheaf. For suitable $\chi$ and $\nu : \mathbb{Q}_\ell \to \mathbb{C}$,

$$\text{Kl}_n(a) = \nu \text{Tr} (\text{Frob}_a, (\text{Kl}^\mathbb{Q}_\ell_n)_a)$$

Deligne: $\text{Kl}^\mathbb{Q}_\ell_n$ is

- concentrated in degree 0 and is a local system
- tamely ramified at 0, maximal unipotent monodromy
- totally wildly ramified at $\infty$, Swan conductor equal to 1
- pure of weight $n - 1$.

Katz: showed that $\text{Kl}^\mathbb{Q}_\ell_n$ is rigid: determined by local monodromies

Gross ($\sim 2010$): $F = \mathbb{F}_q(t)$, automorphic representation for $G(\mathbb{A}_F)$ for all semisimple $G$. For $G = GL_n$, the local representations matched the monodromies calculated by Deligne.
HNY’s automorphic sheaf (a geometric version of Gross’s automorphic representation)

\[ \mathcal{A}_G \text{ on moduli stack of } G\text{-bundles } \text{Bun}_G \text{ on } \mathbb{P}^1 \]

The curve here is \( \mathbb{P}^1_q \supset \mathbb{G}_m \).
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$A_G$ on moduli stack of $G$-bundles $\text{Bun}_G$ on $\mathbb{P}^1$

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Here, $G$ is a non-constant group scheme over $\mathbb{P}^1$, which is isomorphic to $G \times \mathbb{G}_m$ over $\mathbb{G}_m$. Behavior at 0 and $\infty$ encode information about the ramification.
Definition of Kloosterman $D$-module

$$\text{Hecke} := \{(\mathcal{E}_1, \mathcal{E}_2, x \in \mathbb{G}_m, \phi : \mathcal{E}_1|_{\mathbb{P}^1-x} \simeq \mathcal{E}_2|_{\mathbb{P}^1-x})\}.$$ 

Hecke correspondence

$$\begin{array}{ccc}
\text{Hecke} & \xleftarrow{p_1} & \text{Bun}_G \\
\xrightarrow{p_2} & & \text{Bun}_G \times \mathbb{G}_m \\
\end{array}$$

Theorem (Heinloth-Ngo-Yun)

*Heuristic version (actual version uses IC-sheaves):*

$$Rp_2 \ast p_1^* \mathcal{A}_G \cong \mathcal{A}_G \boxtimes Kl_{G^\vee}$$

where $Kl_{G^\vee}$ is the $G^\vee$-Kloosterman sheaf.

Work over $\mathbb{C}$ with $D$-modules to define Kloosterman $D$-modules. For $G^\vee = GL(n)$, recover Deligne’s Kloosterman sheaf.
LG-models appear inside Hecke correspondence

Our idea: a piece of the Hecke correspondence

\[ \text{Hecke} \]

\[ \mathcal{G}_r^r \subset \text{Bun}_G \quad \ast \times \mathcal{G}_m \subset \text{Bun}_G \times \mathcal{G}_m \]

becomes isomorphic to

\[ X = \mathbb{C} \quad \text{after} \]

- basechanging to \( \mathbb{C} \)
- composing with the sum map \( \mathcal{G}_r^r \to \mathcal{G}_a \)
- intersecting with a substack \( \text{Hecke}_\lambda \subset \text{Hecke} \), whose fibers are finite-type \( \text{Gr}_\lambda \subset \text{Gr}_G \).
Other $G^\vee/P^\vee$?

Hodge numbers of CY hypersurfaces $H \subset G^\vee/P^\vee$ vs. exponential hodge numbers of $(X, f)$.

Relation to Langlands functoriality: the quantum connection for $G^\vee/P^\vee$ is naturally a $G^\vee D$-module, even though it is defined as a $\text{GL}(H^*(G^\vee/P^\vee)) D$-module.

For $M$ arbitrary Fano, the quantum connection is a $\text{GL}(H^*(M)) \theta$-connection (Yun, Chen) built from picking a vector $X \in g_1$ in a Vinberg $\theta$-group $g = \bigoplus_{i \in \mathbb{Z}/m\mathbb{Z}} g_i$. It is irregular with slope $1/m$, where $m$ is the Fano index.