Lecture 2: Total positivity and statistical mechanics

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Thanks to Pasha Galashin for some slides!
In the last lecture, I recalled the fundamental connection

\[
\begin{cases}
\text{planar networks with } \\
\text{n sources and n sinks}
\end{cases}
\rightarrow
\begin{cases}
\text{totally nonnegative} \\
\text{n} \times \text{n matrices}
\end{cases}
\]

arising from counting \textit{non-intersecting path families}.
In the last lecture, I recalled the fundamental connection

\[
\left\{ \text{planar networks with } n \text{ sources and } n \text{ sinks} \right\} \rightarrow \left\{ \text{totally nonnegative } n \times n \text{ matrices} \right\}
\]

arising from counting *non-intersecting path families*.

The aim of this lecture is to do an analogous construction for \( \text{Gr}_{\geq 0}(k, n) \).

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In terms of classical enumerative combinatorics, the first two cases are related to enumerating *perfect matchings* and *trees* in graphs. Both are known to be related to determinants: for example, recall the matrix-tree theorem.
Definition (Postnikov (2006))

A point $V \in \text{Gr}(k, n)$ lies in the *totally nonnegative Grassmannian* $\text{Gr}_{\geq 0}(k, n)$ if $\Delta_I(V) \geq 0$ for all $I$.

The *totally positive Grassmannian* $\text{Gr}_{> 0}(k, n)$ is the locus where $\Delta_I(V) > 0$. Example:

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & 3 & 1 & 1 \end{bmatrix} \quad \Delta_{12} = 3 \quad \Delta_{13} = 1 \quad \Delta_{14} = 1 \quad \Delta_{23} = 2 \quad \Delta_{24} = 11 \quad \Delta_{34} = 3$$
Totally nonnegative Grassmannian

**Definition (Postnikov (2006))**

A point \( V \in \text{Gr}(k, n) \) lies in the **totally nonnegative Grassmannian** \( \text{Gr}_{\geq 0}(k, n) \) if \( \Delta_I(V) \geq 0 \) for all \( I \).

The **totally positive Grassmannian** \( \text{Gr}_{> 0}(k, n) \) is the locus where \( \Delta_I(V) > 0 \). Example:

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\begin{bmatrix}
1 & 2 & 0 & -3 \\
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\]

\( \Delta_{12} = 3 \quad \Delta_{13} = 1 \quad \Delta_{14} = 1 \)

\( \Delta_{23} = 2 \quad \Delta_{24} = 11 \quad \Delta_{34} = 3 \)

We have the Plücker relation: for \( \{i_1, \ldots, i_{k-1}\} \) and \( \{j_1, j_2, \ldots, j_{k+1}\} \)

\[
\Delta_{i_1 \cdots i_{k-1} j_1} \Delta_{j_2 \cdots j_{k+1}} - \Delta_{i_1 \cdots i_{k-1} j_2} \Delta_{j_1 j_3 \cdots j_{k+1}} + \cdots + (-1)^k \Delta_{i_1 \cdots i_{k-1} j_{k+1}} \Delta_{j_1 \cdots j_k} = 0
\]

e.g. \( i = 1, \ \{j_1, j_2, j_3\} = \{2, 3, 4\} \)

\( \Delta_{12} \Delta_{34} - \Delta_{13} \Delta_{24} + \Delta_{14} \Delta_{23} = 3 \cdot 3 - 1 \cdot 11 + 2 \cdot 1 = 0. \)
**Dimer model**

- Bipartite graph embedded in a disk, with $n$ boundary vertices.
- Boundary vertices are assumed to have degree one, and by convention we do not draw their colors.
Dimer model

Bipartite graph embedded in a disk, with \( n \) boundary vertices.

Boundary vertices are assumed to have degree one, and by convention we do not draw their colors.

An *almost perfect matching* \( \Pi \) is a collection of edges that uses every interior vertex once, and may use any subset of the boundary vertices.
Dimer model

Bipartite graph embedded in a disk, with \( n \) boundary vertices.

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An almost perfect matching \( \Pi \) is a collection of edges that uses every interior vertex once, and may use any subset of the boundary vertices.

The boundary set \( \partial(\Pi) \) is the set of black boundary vertices used union the set of white boundary vertices not used. \((\partial(\Pi) = \{3, 4\} \text{ in example})\)

We informally call these dimers, that is, polymers consisting of two atoms.

Dimer model: what does a random dimer look like? (Kasteleyn 1967) (Fisher and Temperley 1961)
Boundary measurements

We now assume that $N$ has positive edge weights $w_e$. The weight of an almost perfect matching $\Pi$ is $\text{wt}(\Pi) = \prod_{e \in \Pi} w_e$.

**Definition (Dimer generating function)**

For a subset $I \subset [n]$, define the *boundary measurement* $\Delta_I(N) := \sum_{\Pi: \partial(\Pi) = I} \text{wt}(\Pi)$

If almost perfect matchings exist, it is easy to see that there is a unique value of $k$ such that $\Delta_I(N) \neq 0$ only if $|I| = k$. 
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\[\begin{align*}
\Delta_{12}(N) &= a & \Delta_{23}(N) &= d \\
\Delta_{13}(N) &= ac + bd & \Delta_{24}(N) &= 1 \\
\Delta_{14}(N) &= b & \Delta_{34}(N) &= c
\end{align*}\]
Boundary measurements

We now assume that \( N \) has positive edge weights \( w_e \). The \textit{weight} of an almost perfect matching \( \Pi \) is \( \text{wt}(\Pi) = \prod_{e \in \Pi} w_e \).

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\end{align*}
\]
Boundary measurement map

Theorem (Postnikov, Talaska, Postnikov-Speyer-Williams, Kuo, L.)

The map \( N \to (\Delta_I(N))_{I \in \binom{[n]}{k}} \) defines a point \( M(N) \in \text{Gr}(k, n) \).

To prove the theorem, it suffices to check that \( \Delta_I(N) \) satisfies the Plücker relations.

Theorem (Postnikov)

1. The map \( N \to M(N) \) surjects onto \( \text{Gr}_{\geq 0}(k, n) \).
2. If \( M(N) = M(N') \), then \( N \) and \( N' \) are related by local moves.
3. For each positroid cell \( \Pi_{f, > 0} \), there exists a network \( N(t_1, t_2, \ldots, t_d) \) with edge weights given by the parameters \( t_1, \ldots, t_d \) such that the map \( (t_1, t_2, \ldots, t_d) \in \mathbb{R}^d_{>0} \to M(N(t_1, t_2, \ldots, t_d)) \) is a homeomorphism \( \mathbb{R}^d_{>0} \cong \Pi_{f, > 0} \).
Since \( \Delta_I(N) \Delta_J(N) \) counts \textit{double dimers}, the Plücker relation is equivalent to a statement about boundary connections of double dimers.
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![Diagram](image_url)
Since $\Delta_I(N)\Delta_J(N)$ counts double dimers, the Plücker relation is equivalent to a statement about boundary connections of double dimers.

For a $(k, n)$-partial noncrossing matching $\tau$,

$$F_\tau(N) = \sum \text{wt}(\Sigma)$$

summed over double dimers $\Sigma$ with connectivity $\tau$. 

Double dimers

Since $\Delta_I(N)\Delta_J(N)$ counts **double dimers**, the Plücker relation is equivalent to a statement about boundary connections of double dimers.


For a $(k,n)$-**partial noncrossing matching** $\tau$,

$$F_\tau(N) = \sum \text{wt}(\Sigma)$$

summed over double dimers $\Sigma$ with connectivity $\tau$.

We have an identity $\Delta_I \Delta_J = \sum F_\tau$, summed over $\tau$ compatible with $(I,J)$.
Let $G$ be a planar bipartite graph. Then for any positive edge weights $w_e$, we have

$$M((G, w_e)) \in \Pi_{f, >0}$$

where $f = f_G$ only depends on the underlying graph $G$. 
Let $G$ be a planar bipartite graph. Then for \textit{any} positive edge weights $w_e$, we have

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- We have $\dim(\Pi_{f,>0}) \leq \#\text{Faces}(G) - 1$, and when equality holds we call $G$ reduced.
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- We have $\dim(\Pi_{f, >0}) \leq \#\text{Faces}(G) - 1$, and when equality holds we call $G$ reduced.
- In the reduced case, we can read $f$ off of $G$ by the rules of the road.
(1) = 4
\[ f(1) = 4 \]
$f(1) = 4$
An electrical resistor network is an undirected weighted graph $\Gamma$.

Edge weight = conductance = $1/$resistance

Some vertices are designated as boundary vertices. The rest are interior vertices.
The electrical properties are described by the \textit{response matrix} \( \Lambda(\Gamma) \):

\[
\Lambda(\Gamma) : \mathbb{R}^{\#\text{boundary vertices}} \rightarrow \mathbb{R}^{\#\text{boundary vertices}}
\]

voltage vector \( \rightarrow \) current vector

which gives the current that flows through the boundary vertices when specified voltages are applied.
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voltage vector \( \mapsto \) current vector

which gives the current that flows through the boundary vertices when specified voltages are applied.

\[ \Lambda_{ij} = \text{current flowing through vertex } j \text{ when the voltage is set to 1 at vertex } i \text{ and 0 at all other vertices.} \]

Possibly surprisingly, \( \Lambda(\Gamma) \) is a symmetric matrix. If all vertices are considered boundary vertices, then \( \Lambda(\Gamma) \) is simply the \textit{Laplacian matrix} of \( \Gamma \).
Axioms of electricity

The matrix $\Lambda(\Gamma)$ can be computed using only two axioms.

**Kirchhoff’s Law (1845)**

The sum of currents flowing into an interior vertex is equal to 0.

Ohm’s Law (1827)

For each resistor we have

$$V_1 - V_2 = I \times R$$

where $I =$ current flowing through the resistor, $V_1$, $V_2 =$ voltages at two ends of resistor, $R =$ resistance of the resistor.
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**Ohm’s Law (1827)**

For each resistor we have

$$(V_1 - V_2) = I \times R$$

where

$I = \text{current flowing through the resistor}$

$V_1, V_2 = \text{voltages at two ends of resistor}$

$R = \text{resistance of the resistor}$

To compute $\Lambda(\Gamma)$, we give variables to each edge (current through that edge) and each vertex (voltage at that vertex). Then solve a large system of linear equations.
We now assume that $\Gamma$ is embedded into a disk. A *grove* $F$ in $\Gamma$ is a subforest such that every interior vertex is connected to some boundary vertex.

The boundary partition $\sigma(F)$ of a grove $F$ is the noncrossing partition whose parts are boundary vertices belonging to the same component of $F$.

\[
\sigma(F) = \{2, 3, 4 | 1, 5\}
\]

Planarity $\Rightarrow$ noncrossing.

Groves were studied by Carroll–Speyer, Kenyon–Wilson, ...

Thomas Lam (U.Michigan, IAS)
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![Diagram showing a grove with boundary vertices 1, 2, 3, 4, 5 and the corresponding boundary partition]
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Planarity $\iff$ noncrossing.

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The noncrossing partition $\sigma = \{1, 2, 5, 9|3, 4|6, 7, 8|10, 11|12\}$. 
The noncrossing partition $\sigma = \{1, 2, 5, 9|3, 4|6, 7, 8|10, 11|12\}$. Let $\mathcal{NC}_n$ denote the set of noncrossing partitions on $\{1, \ldots, n\}$. Then $|\mathcal{NC}_n| = C_n = \frac{1}{n+1} \binom{2n}{n}$. For $n = 3$, we have 5 noncrossing partitions.

$$(123), (1|23), (12|3), (13|2), (1|2|3).$$
Definition (Grove generating function)

For $\sigma \in \mathcal{NC}_n$, and an electrical network $\Gamma$, define

$$L_{\sigma}(\Gamma) = \sum_{\sigma(F) = \sigma} \text{wt}(F)$$

where the weight of a grove $F$ is the product of the weights of the edges belonging to $F$.
Grove measurements

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We collect all the $L_{\sigma}$’s together to obtain a map

$$\Gamma \mapsto \mathcal{L}(\Gamma) = (L_{\sigma}(\Gamma))_{\sigma \in \mathcal{NC}_n} \in \mathbb{P}^{\mathcal{NC}_n}.$$
Grove measurements

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**Proposition (essentially Kirchhoff 1800s)**

$\Lambda(\Gamma) = \Lambda(\Gamma')$ if and only if $L(\Gamma) = L(\Gamma')$
Grove measurements

Definition (Grove generating function)

For $\sigma \in \mathcal{NC}_n$, and an electrical network $\Gamma$, define

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Proposition (essentially Kirchhoff 1800s)

$\Lambda(\Gamma) = \Lambda(\Gamma')$ if and only if $\overrightarrow{L}(\Gamma) = \overrightarrow{L}(\Gamma')$

Define the **compactified space of circular planar electrical networks**:

$$\mathcal{E}_n := \left\{ \overrightarrow{L}(\Gamma) \mid \Gamma \text{ planar electrical network} \right\} \subset \mathbb{P}^{\mathcal{NC}_n}$$
Example: the grove embedding

\[ L_{1|2|3} = a + b + c, \]
\[ L_{123} = abc \]
\[ L_{12|3} = ab, \]
\[ L_{1|23} = bc, \]
\[ L_{13|2} = ac, \]

\[ \mathcal{L}(\Gamma) = (a + b + c : ab : bc : ac : abc) \in \mathbb{P}^4 \]
Consider the (degenerate) skew-symmetric bilinear form on $\mathbb{R}^{2n}$

$$
\langle x, y \rangle = \sum_{k=1}^{2n} (-1)^k (x_k y_{k+1} - x_{k+1} y_k)
$$

where $x_{2n+1} = (-1)^n x_1$. A subspace $U \subset \mathbb{R}^{2n}$ is *isotropic* if $\langle \cdot, \cdot \rangle$ restricts to 0 on $U$. We set

$$
\text{LG}(n+1, 2n) := \{ U \subset \mathbb{R}^{2n} \mid U \text{ is maximal isotropic} \} \subset \text{Gr}(n+1, 2n).
$$

We have $\dim(\text{Gr}(n+1, 2n)) = n^2 - 1$ but $\dim \text{LG}(n+1, 2n) = n(n-1)/2$. 
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**Definition**

The **totally nonnegative Lagrangian Grassmannian**:

$$
\text{LG}_{\geq 0}(n+1, 2n) := \text{LG}(n+1, 2n) \cap \text{Gr}_{\geq 0}(n+1, 2n).
$$

Our notion differs from that of Lusztig and Karpman. (Thanks to David Speyer for a helpful discussion!)
There is a homeomorphism

\[ \iota : \mathcal{E}_n \longrightarrow LG_{\geq 0}(n + 1, 2n) \]

given by the formula

\[ \Delta_I(\iota(\Gamma)) = \sum_{\sigma \in NC_n} a_{I\sigma} L_\sigma(\Gamma) \]

where \(a_{I\sigma}\) is a 0-1 matrix, with the 1-s given by concordant pairs \((I, \sigma)\).

The Ising model is a model for ferromagnetism. (Lenz 1920, Ising 1925)

$G =$ planar network in a disk (boundary vertices may have deg $> 1$)

$J_e =$ weight of edge $e$
Ising model

\( G = \) planar network in a disk (boundary vertices may have \( \text{deg} > 1 \))

\( J_e = \) weight of edge \( e \)

Spin configuration: a map \( \sigma : V \rightarrow \{ \pm 1 \} \)

\[ \text{wt}(\sigma) := \prod_{\{u,v\} \in E} \exp \left( J_{\{u,v\}} \sigma_u \sigma_v \right) \]
Ising model

\[ G = \text{planar network in a disk (boundary vertices may have deg} \geq 1) \]

\[ J_e = \text{weight of edge } e \]

**Spin configuration:** a map \( \sigma : V \to \{\pm 1\} \)

\[
\begin{align*}
\text{wt}(\sigma) &:= \prod_{\{u,v\} \in E} \exp \left( J_{\{u,v\}} \sigma_u \sigma_v \right) \\
\text{wt}(\sigma) &= \frac{\exp \left( J_{e_1} + J_{e_2} + J_{e_6} + J_{e_8} \right)}{\exp \left( J_{e_3} + J_{e_4} + J_{e_5} + J_{e_7} + J_{e_9} \right)} \\
\text{Prob}(\sigma) &:= \frac{\text{wt}(\sigma)}{Z}
\end{align*}
\]
The Ising model is a model for ferromagnetism. (Lenz 1920, Ising 1925)
Correlation: $\langle \sigma_u \sigma_v \rangle := \text{Prob}(\sigma_u = \sigma_v) - \text{Prob}(\sigma_u \neq \sigma_v)$.

**Definition**

Boundary correlation matrix: $M(G, J) = (m_{ij})_{i,j=1}^n$, where $m_{ij} := \langle \sigma_{b_i} \sigma_{b_j} \rangle$.
Boundary correlations I

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Lives inside $\mathbb{R}^{n \choose 2}$.
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\[ M(G, J) \] is a symmetric matrix with 1’s on the diagonal and nonnegative entries. Lives inside \[ \mathbb{R}^{n \choose 2} \]

\[ \mathcal{X}_n := \{ M(G) \mid G \text{ is a planar network with } n \text{ boundary vertices} \} \]
\[ \overline{\mathcal{X}}_n := \text{closure of } \mathcal{X}_n \text{ inside the space of } n \times n \text{ matrices} \]
\[ M(G) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \quad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2 \exp(J_e) - 2 \exp(-J_e)}{2 \exp(J_e) + 2 \exp(-J_e)}. \]
Boundary correlations II

\[ M(G) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \quad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2 \exp(J_e) - 2 \exp(-J_e)}{2 \exp(J_e) + 2 \exp(-J_e)} \]

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We have \( \mathcal{X}_2 \cong [0, 1] \) and \( \overline{\mathcal{X}}_2 \cong [0, 1] \).
Boundary correlations II

\[ M(G) = \begin{pmatrix} 1 & m_{12} \\ m_{12} & 1 \end{pmatrix}, \quad m_{12} = \langle \sigma_1 \sigma_2 \rangle = \frac{2 \exp(J_e) - 2 \exp(-J_e)}{2 \exp(J_e) + 2 \exp(-J_e)} \]

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- We have $\mathcal{X}_2 \cong [0, 1)$ and $\overline{\mathcal{X}}_2 \cong [0, 1]$.
- $\overline{\mathcal{X}}_n$ is obtained from $\mathcal{X}_n$ by allowing $J_e = \infty$ (i.e., contracting edges).
Consider the symmetric nondegenerate bilinear form on \( \mathbb{R}^{2n} \) given by

\[
(x, y) = \sum_{i=1}^{2n} (-1)^i x_i y_i.
\]

A subspace \( W \subset \mathbb{R}^{2n} \) is *isotropic* if the restriction of \((\cdot, \cdot)\) to \( W \) is identically 0. The *orthogonal Grassmannian* is given by

\[
\text{OG}(n, 2n) := \{ W \in \text{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I \}\]

and consists of a component of the isotropic subspaces of \( \text{Gr}(n, 2n) \). We have \( \text{dim}(\text{Gr}(n, 2n)) = n^2 \) but \( \text{dim}(\text{OG}(n, 2n)) = \binom{n}{2} = \frac{n(n-1)}{2} \).
Consider the symmetric nondegenerate bilinear form on $\mathbb{R}^{2n}$ given by

$$(x, y) = \sum_{i=1}^{2n} (-1)^i x_i y_i.$$ 

A subspace $W \subset \mathbb{R}^{2n}$ is isotropic if the restriction of $(\cdot, \cdot)$ to $W$ is identically 0. The orthogonal Grassmannian is given by

$${\text{OG}}(n, 2n) := \{ W \in \text{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \text{ for all } I \}$$

and consists of a component of the isotropic subspaces of $\text{Gr}(n, 2n)$. We have $\dim(\text{Gr}(n, 2n)) = n^2$ but $\dim({\text{OG}}(n, 2n)) = \binom{n}{2} = \frac{n(n-1)}{2}$.

**Definition (Huang–Wen)**

The totally nonnegative orthogonal Grassmannian:

$${\text{OG}}_{\geq 0}(n, 2n) := {\text{OG}}(n, 2n) \cap \text{Gr}_{\geq 0}(n, 2n), \text{ i.e.,}$$

$${\text{OG}}_{\geq 0}(n, 2n) := \{ W \in \text{Gr}(n, 2n) \mid \Delta_I(W) = \Delta_{[2n]\setminus I}(W) \geq 0 \text{ for all } I \}.$$ 

This notion differs from a general one of Lusztig.
\[ \mathcal{X}_n := \{ M(G) \mid G \text{ is a planar Ising network with } n \text{ boundary vertices} \} \]

\[ \overline{\mathcal{X}}_n := \text{closure of } \mathcal{X}_n \text{ inside the space of } n \times n \text{ matrices.} \]

We have \[ \mathcal{X}_n, \overline{\mathcal{X}}_n \subset \text{Mat}^{\text{sym}}_n(\mathbb{R}, 1) := \left\{ \begin{array}{l}
\text{symmetric } n \times n \text{ matrices} \\
\text{with 1’s on the diagonal}
\end{array} \right\}. \]
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The doubling map \( \phi: \)

\[
\begin{pmatrix}
1 & m_{12} & m_{13} & m_{14} \\
m_{12} & 1 & m_{23} & m_{24} \\
m_{13} & m_{23} & 1 & m_{34} \\
m_{14} & m_{24} & m_{34} & 1
\end{pmatrix} \mapsto \begin{pmatrix}
1 & 1 & m_{12} & -m_{12} & -m_{13} & m_{13} & m_{14} & -m_{14} \\
-m_{12} & m_{12} & 1 & 1 & m_{23} & -m_{23} & -m_{24} & m_{24} \\
m_{13} & -m_{13} & -m_{23} & m_{23} & 1 & 1 & m_{34} & -m_{34} \\
-m_{14} & m_{14} & m_{24} & -m_{24} & -m_{34} & m_{34} & 1 & 1
\end{pmatrix}
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Theorem (Galashin–Pylyavskyy (2018))

The map \( \phi \) restricts to a homeomorphism between \( \mathcal{X}_n \) and \( \mathcal{OG} \geq 0(n, 2n) \).
\[\mathcal{X}_n := \{ M(G) \mid G \text{ is a planar Ising network with } n \text{ boundary vertices} \}\]

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- m_{14} & - m_{24} & - m_{34} & 1
\end{pmatrix}
\rightarrow
\begin{pmatrix}
1 & 1 & m_{12} & - m_{12} & - m_{13} & m_{13} & m_{14} & - m_{14} \\
- m_{12} & m_{12} & 1 & 1 & m_{23} & - m_{23} & - m_{24} & m_{24} \\
m_{13} & - m_{13} & - m_{23} & m_{23} & 1 & 1 & m_{34} & - m_{34} \\
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\end{pmatrix}
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Theorem (Galashin–Pylyavskyy (2018))

The map $\phi$ restricts to a homeomorphism between $\mathcal{X}_n$ and $\mathcal{O}_G(G \geq 0(n, 2n))$.

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The doubling map $\phi$:

$$
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- m_{12} & 1 & m_{23} & m_{24} \\
- m_{13} & - m_{23} & 1 & m_{34} \\
- m_{14} & - m_{24} & - m_{34} & 1
\end{pmatrix} \mapsto 
\begin{pmatrix}
1 & 1 & m_{12} & - m_{12} & - m_{13} & m_{13} & m_{14} & - m_{14} \\
- m_{12} & 1 & 1 & m_{23} & - m_{23} & - m_{24} & m_{4} & - m_{4} \\
- m_{13} & - m_{13} & - m_{23} & m_{34} & 1 & 1 & m_{34} & - m_{34} \\
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\end{pmatrix}
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**Theorem (Galashin–Pylyavskyy (2018))**

The map \( \phi \) restricts to a homeomorphism between \( \overline{\mathcal{X}}_n \) and \( \text{OG}_{\geq 0}(n, 2n) \).
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**Theorem (Galashin–Pylyavskyy (2018))**

The map \( \phi \) restricts to a homeomorphism between \( \overline{\mathcal{X}}_n \) and \( \text{OG}_{\geq 0}(n, 2n) \).

\( \overline{\mathcal{X}}_n \xrightarrow{\phi} \text{OG}_{\geq 0}(n, 2n) \)

Lis (2016): boundary correlations related to \( \text{Gr}_{\geq 0}(n, 2n) \).
The electrical network model and Ising model have the same indexing set for strata, same closure relations, and same local moves (on the level of unweighted graphs).
Electrical network $\rightarrow$ planar bipartite graph
Ising network $\rightarrow$ planar bipartite graph

Here $s_e := \text{sech}(2J_e)$, $c_e := \tanh(2J_e)$ so that $s_e^2 + c_e^2 = 1$. 
Uncrossing partial order $P_n$

Let $\hat{P}_n$ be $P_n$ with a minimum $\hat{0}$ added. $\hat{P}_n$ is Eulerian (L.) $\hat{P}_n$ is shellable (Kenyon–Hersh)

\[
\begin{array}{c}
\times \\
\rightarrow \begin{array}{c}
\lessgtr \\
\text{or} \\
\langle \rangle
\end{array}
\end{array}
\]
Let $\hat{P}_n$ be $P_n$ with a minimum $\hat{0}$ added.

- $\hat{P}_n$ is Eulerian (L.)
- $\hat{P}_n$ is shellable (Kenyon–Hersh)
Further directions

- Explain the surprising similarity between the combinatorics appearing in electrical networks and that in Ising models.
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Some references:
- T. Lam, Totally nonnegative Grassmannian and Grassmann polytopes, CDM lectures 2014.
- T. Lam, Electroid varieties and a compactification of the space of planar electrical networks, Adv. in Math. 2018.