

Math 463: Introduction to Mathematical Biology

Homework #1 : Linear Difference Equations Solutions

Chapter 1

- 2a.** (6 points) Solve the following difference equations subject to the specified initial conditions and sketch the solution: $x_n - 5x_{n-1} + 6x_{n-2} = 0$; $x_0 = 2, x_1 = 5$
Characteristic Equation (1 point): $\lambda^2 - 5\lambda + 6 = 0$
Eigenvalues (1 point): $\lambda_1 = 3, \lambda_2 = 2$.
General Solution (1 point): $x_n = A_1 3^n + A_2 2^n$, for unknown constants A_1, A_2 .
Apply the initial conditions (1 point):

$$\begin{aligned} 2 &= A_1 + A_2 \\ 5 &= 3A_1 + 2A_2, \end{aligned}$$

which yields a particular solution (1 point): $x_n = 3^n + 2^n$.
Graph (1 point):

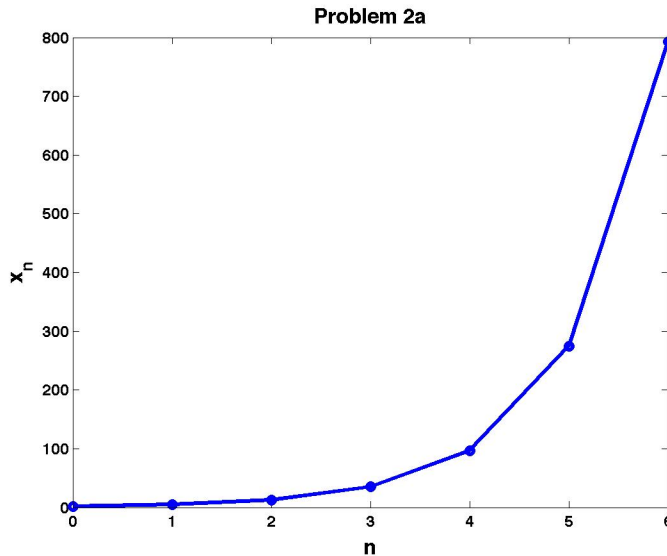


Figure 1: Plot of $x_n = 3^n + 2^n$ as a function of n .

- 2e.** (6 points) Solve the following difference equations subject to the specified initial conditions and sketch the solution: $x_{n+2} + x_{n+1} - 2x_n = 0$; $x_0 = 6, x_1 = 3$
 Characteristic Equation (1 point): $\lambda^2 + \lambda - 2 = 0$
 Eigenvalues (1 point): $\lambda_1 = -2, \lambda = 1$.
 General Solution (1 point): $x_n = A_1(-2)^n + A_2(1)^n$, for unknown constants A_1, A_2 .
 Apply the initial conditions (1 point):

$$\begin{aligned} 6 &= A_1 + A_2 \\ 3 &= -2A_1 + A_2, \end{aligned}$$

which yields a particular solution (1 point): $x_n = 5 + (-2)^n$.

Graph (1 point):

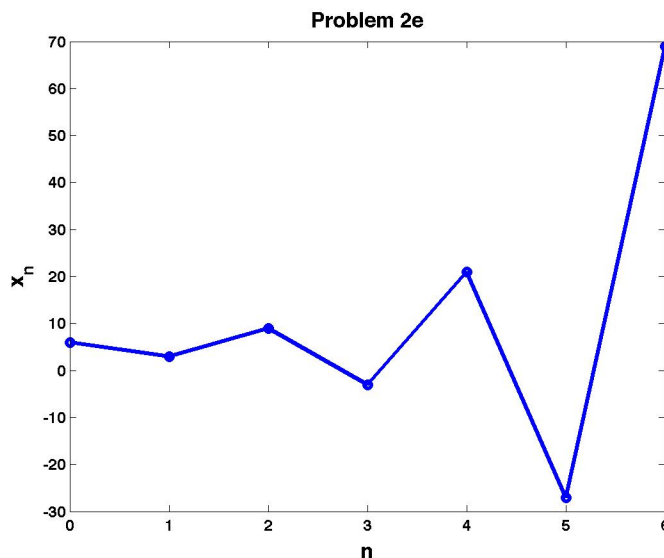


Figure 2: Plot of $x_n = 5^n + (-2)^n$ as a function of n .

- 6a.** (8 points) Given the two first order equations

$$\begin{aligned} x_{n+1} &= 3x_n + 2y_n \\ y_{n+1} &= x_n + 4y_n, \end{aligned}$$

the higher order system is (2 points): $x_{n+2} - 7x_{n+1} + 10x_n = 0$.
 Characteristic Equation (1 point): $\lambda^2 - 7\lambda + 10 = 0$
 Eigenvalues (1 point): $\lambda_1 = 5, \lambda_2 = 2$.
 General Solution (1 point): $x_n = A_15^n + A_22^n$.
 Graph (2 points):

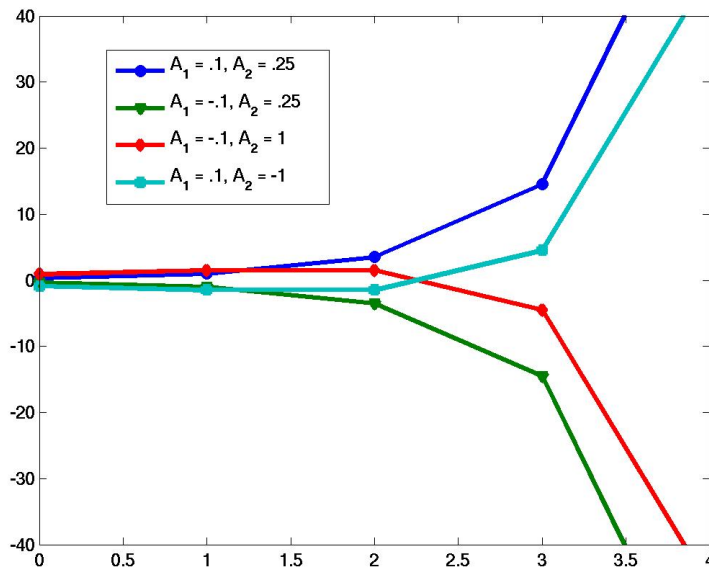


Figure 3: Plot of $x_n = A_1 5^n + A_2 (2)^n$ as a function of n for various choices of the constants, A_1 and A_2 .

Discussion (1 point) As you can see, different initial conditions yield different values for A_1 and A_2 , but eventually the iterations will diverge to positive or negative infinity depending on the sign of A_1 .

6b. (8 points): Given the two first order equations

$$\begin{aligned} x_{n+1} &= \frac{1}{4}x_n + y_n \\ y_{n+1} &= \frac{3}{16}x_n - \frac{1}{4}y_n, \end{aligned}$$

the higher order system is (2 points): $x_{n+2} - \frac{1}{4}x_n = 0$.

Characteristic Equation (1 point): $\lambda^2 - \frac{1}{4} = 0$

Eigenvalues (1 point): $\lambda_1 = \frac{1}{2}, \lambda_2 = -\frac{1}{2}$.

General Solution (1 point): $x_n = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{2}\right)^n$.

Graph (2 points):

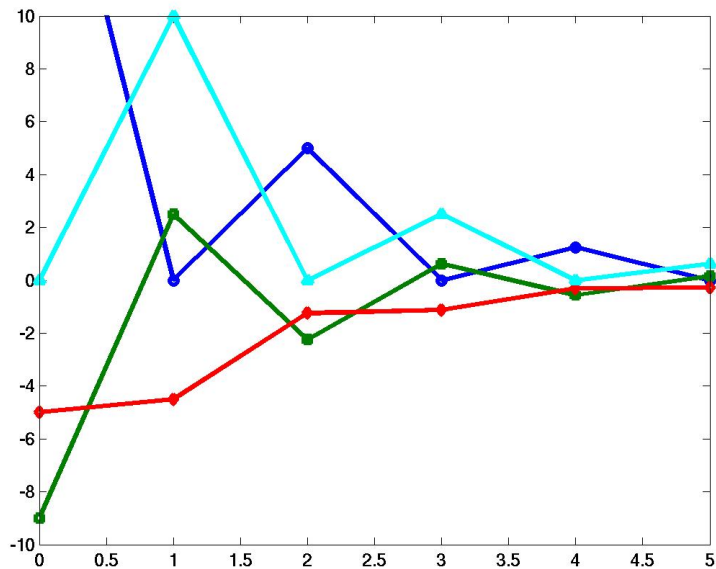


Figure 4: Plot of $x_n = A_1 \left(\frac{1}{2}\right)^n + A_2 \left(-\frac{1}{2}\right)^n$ as a function of n for various choices of the constants, A_1 and A_2 .

Discussion (1 point): As you can see, different initial conditions yield different values for A_1 and A_2 , but the iterations still oscillate, as they decay toward zero.

9a. (6 points) Solve and graph the solutions of: $x_{n+2} + x_n = -0$.

Characteristic Equation (1 point): $\lambda^2 + \lambda = 0$

Eigenvalues (1 point): $\lambda = \pm i$.

General Solution (2 point):

$$\begin{aligned}
 x_n &= A_1 i^n + A_2 (-i)^n \\
 &= A_1 (e^{i\frac{\pi}{2}})^n + A_2 (e^{-i\frac{\pi}{2}})^n \\
 &= C_1 \cos \frac{n\pi}{2} + C_2 \sin \frac{n\pi}{2}
 \end{aligned}$$

for unknown constants C_1 and C_2 (or A_1 and A_2).

Graph (2 points):

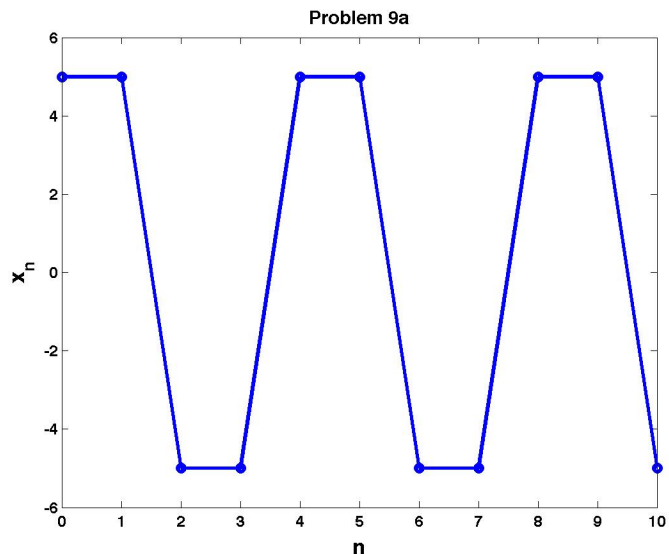


Figure 5: Plot of $x_n = A_1 \cos \frac{n\pi}{2} + A_2 \sin \frac{n\pi}{2}$ as a function of n for constants, $A_1 = A_2 = 5$.

Problem 15 (11 points)

Given

a_n = number of terminal segments

b_n = number of next-to-terminal segments

s_n = total number of segments

15a. (6 points) The number of terminal segments in the next generation (ie the $n+1$ generation) is equal to the number of terminal segments in the previous generation that produced a single daughter (pa_n) plus twice the number of terminal segments in the previous generation that produced two daughters ($2qa_n$) plus the number of next-to-terminal segments in the previous generation that produced a single daughter. The number of next-to-terminal segments in the next generation is equal to the number of terminal segments in the previous generation because all terminal segments grow, thereby becoming next-to-terminal segments. Finally the total number of segments in the next generation is equal to the total number of segments in the previous generation plus the number current number of terminal segments. Alternatively, the total number of segments in the next generation is equal to the sum of the terminal segments from all previous generations. This leads to the following system of equations

$$a_{n+1} = pa_n + 2qa_n + rb_n$$

$$b_{n+1} = a_n$$

$$s_{n+1} = s_n + a_{n+1} \quad \text{or} \quad s_{n+1} = \sum_{j=1}^{n+1} a_j$$

15b. (1 point) Reduce this system to a single higher order equation by substituting

a_{n-1} for b_n and making use of that fact the $p + q = 1$.

$$\begin{aligned} a_{n+1} &= (1 - q)a_n + 2qa_n + ra_{n-1} \\ a_{n+1} - (1 + q)a_n - ra_{n-1} &= 0 \end{aligned}$$

15c. (5 points) Given $a_0 = 1$ and $a_1 = 1 + q$
 Characteristic Equation: $\lambda^2 - (1 + q)\lambda - r = 0$
 Eigenvalues: $\lambda = \frac{1}{2}\{1 + q \pm \sqrt{(1 + q)^2 + 4r}\}$
 Solution: $a_n = A_1\lambda_1^n + A_2\lambda_2^n$.
 The initial condition then give us that

$$\begin{aligned} A_1 &= \frac{\lambda_1}{\lambda_1 - \lambda_2} \\ A_2 &= -\frac{\lambda_2}{\lambda_1 - \lambda_2} \end{aligned}$$

and that the general solution is $a_n = \frac{\lambda_1}{\lambda_1 - \lambda_2}\lambda_1^n - \frac{\lambda_2}{\lambda_1 - \lambda_2}\lambda_2^n$.

After 10 days:

Terminal segments: $a_{10} = \frac{\lambda_1}{\lambda_1 - \lambda_2}\lambda_1^{10} - \frac{\lambda_2}{\lambda_1 - \lambda_2}\lambda_2^{10}$

Total number of segments: $s_{10} = \sum_{j=1}^{10} \left(\frac{\lambda_1}{\lambda_1 - \lambda_2}\lambda_1^j - \frac{\lambda_2}{\lambda_1 - \lambda_2}\lambda_2^j \right)$

Problem 16 (5 points) Original system of two first order equations:

$$\begin{aligned} R_{n+1} &= (1 - f)R_n + m_n \\ m_{n+1} &= \gamma f R_n \end{aligned}$$

16a. (1 point) $R_{n+1} = (1 - f)R_n + \gamma f R_{n-1}$

16b. (3 points)

Characteristic Equation: $\lambda^2 - (1 - f)\lambda - \gamma f = 0$.

Eigenvalues: $\lambda = \frac{1}{2}\{1 - f \pm \sqrt{(1 - f)^2 + 4\gamma f}\}$.

Magnitude: If we designate $\lambda_1 > 0$ and $\lambda_2 < 0$, λ_1 dominates because $\lambda_1 > |-\lambda_2|$.

16c. (1 point) If $\lambda_1 = 1$, then

$$\begin{aligned} 1 &= \frac{1 - f}{2} + \frac{\sqrt{(1 - f)^2 + 4\gamma f}}{2} \\ (1 + f)^2 &= (1 - f)^2 + 4\gamma f \\ 0 &= 4f(\gamma - 1) \end{aligned}$$

and $\gamma = 1$.

16d. (1 point) If $\lambda_1 = \gamma = 1$, then $\lambda_2 = -f$ and thus $|\lambda_2| < 1$. Since $R_n = A_1 + A_2(-f)^n$, the solutions will then exhibit oscillations (centered at A_1) which decrease in magnitude.

Problem 18 (20 points) 18a. (1 point) Given the system

$$\begin{aligned} C_{n+1} &= C_n - \beta V_n + m \\ V_{n+1} &= \alpha C_n, \end{aligned}$$

a simple substitution yields $C_{n+1} = C_n - \beta\alpha C_{n-1} + m$.

18b. (4 points) We now look for a particular solution of the form $C_p = \kappa$. The substitution of C_p into the second order equation yields $\kappa = m/(\alpha\beta)$.

To solve for the homogeneous solution:

Characteristic Equation: $\lambda^2 - \lambda + \alpha\beta = 0$.

Eigenvalues: $\lambda = \frac{1}{2}\{1 \pm \sqrt{1 - 4\alpha\beta}\}$.

Homogeneous solution: $(C_H)_n = A_1\lambda_1^n + A_2\lambda_2^n$ for undefined constants A_1 and A_2 .

General solution:

$$\begin{aligned} C_n &= A_1\lambda_1^n + A_2\lambda_2^n + \frac{m}{\alpha\beta} \\ &A_1\left\{\frac{1}{2} + \frac{1}{2}\sqrt{1 - 4\alpha\beta}\right\}^n + A_2\left\{\frac{1}{2} - \frac{1}{2}\sqrt{1 - 4\alpha\beta}\right\}^n + \frac{m}{\alpha\beta} \end{aligned}$$

18c1. (5 points) If $4\alpha\beta < 1$, then $\beta < \frac{1}{4\alpha}$

Interpretation: The amount of CO_2 lost per breath is less than a quarter of the “amount of CO_2 that induces a unit volume of breathing.”

Under this assertion (note that there is also the implicit assumption that $\alpha, \beta > 0$), we want evidence that the iterations tend toward $m/(\alpha\beta)$. To show this we, must first find the dominant eigenvalue.

Clearly if $4\alpha\beta < 1$, both eigenvalues are real and positive. Therefore if $\lambda_1 = \frac{1}{2}\{1 + \sqrt{1 - 4\alpha\beta}\}$, we know that

$$|\lambda_1| > |\lambda_2|$$

Now that we know λ_1 is the dominant eigenvalue, we need to show that $|\lambda_1| < 1$. So we do the following:

$$\begin{aligned} \lambda_1 &= \frac{1}{2}\{1 + \sqrt{1 - 4\alpha\beta}\} \\ 2\lambda_1 &= 1 + \sqrt{1 - 4\alpha\beta} \\ (2\lambda_1 - 1)^2 &= 1 - 4\alpha\beta \\ 0 < 1 - (2\lambda_1 - 1)^2 &= 4\alpha\beta < 1 \\ 0 < (2\lambda_1 - 1)^2 &< 1 \\ \frac{1}{2} < |\lambda_1| &< 1 \end{aligned}$$

So we can thus conclude that

$$1 > |\lambda_2| > |\lambda_1|$$

and therefore the iterations decay to a constant equilibrium steady state C_e .

If we set $C_n = C_e$ and solve for C_e , we find

$$\begin{aligned} C_e - C_e + \alpha\beta C_e &= m \\ C_e &= \frac{m}{\alpha\beta} \\ V_e &= \alpha C_e = \frac{m}{\beta}. \end{aligned}$$

17d. (5points) If $4\alpha\beta > 1$, then $\lambda = \frac{1}{2}\{1 \pm \sqrt{1 - 4\alpha\beta}\} = \frac{1}{2}\{1 \pm i\sqrt{4\alpha\beta - 1}\}$ will be complex so the CO₂ levels will undergo oscillations.

Let $\gamma = \sqrt{4\alpha\beta - 1}$ and thus $\lambda_1 = \frac{1}{2} + i\frac{\gamma}{2}$ and $\lambda_2 = \frac{1}{2} - i\frac{\gamma}{2}$.

Consider, the situation for which the amplitude of the oscillations will be constant.

That is when $r = |\lambda_{1,2}| = \sqrt{a^2 + b^2} = 1$.

$$\begin{aligned} |\lambda_{1,2}| &= \sqrt{\frac{1}{4} + \frac{\gamma^2}{4}} \stackrel{?}{=} 1 \\ &\sqrt{1 + \gamma^2} \stackrel{?}{=} 2 \\ &1 + \gamma^2 \stackrel{?}{=} 2 \\ &\gamma \stackrel{?}{=} \sqrt{3} \\ &\sqrt{4\alpha\beta - 1} \stackrel{?}{=} \sqrt{3} \\ &4\alpha\beta - 1 \stackrel{?}{=} 3 \\ &4\alpha\beta \stackrel{?}{=} 4. \end{aligned}$$

Since we previously assumed that $4\alpha\beta > 1$, we then know that $|\lambda_{1,2}| > 1$, which yields iterations which simultaneously oscillate and increase in magnitude.

The frequency of the oscillations will be $\arctan(\gamma)$ (via Euler's formula). In the case where $4\alpha\beta > 1$, we know that $\gamma > \sqrt{3}$, thus the frequency of oscillation must be greater than $\pi/3$.

17d. One could make the solution more realistic by claiming that sensitivity to blood CO₂ has some maximum response capacity

$$V_{n+1} = \alpha C_n (1 - C_n / C_{max}),$$

which would yield the equation

$$C_{n+1} - C_n + \alpha\beta C_{n-1}(1 - C_{n-1}/C_{max}) = m.$$

This equation has two steady states (can you find them? are they both physically reasonable?).