Math 463: Introduction to Mathematical Biology

Homework 4 Solutions

The logistic growth model takes the following form:

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right), \qquad N(0) = N_0. \tag{1}$$

1. The logistic growth model is a phenomenological model that has a long list of assumptions which underlie it. List a few.

2(a). In order to find and analytic expression for the solution of equation (1), separate the dependent and independent variables and integrate.

$$\int \frac{dN}{N\left(1 - \frac{N}{K}\right)} = \int r dt$$

Use partial fractions to rewrite the left-hand side

$$\int \frac{dN}{N} + \int \frac{dN}{K - N} = \int r dt$$

This yields $\log N - \log K - N = rt + C$. Solving this for N gives

$$N(t) = \frac{KCe^{rt}}{1 + Ce^{rt}}$$

Now use the initial condition to solve for C. You should find $C = \frac{N_0}{K - N_0}$. Plugging this into the equation above results in

$$N(t) = \frac{KN_0}{N_0 + (K - N_0)e^{-rt}}$$

2(b). The fact that all $N(t)$ will always approach K, independent of the initial condition can be seen by taking the following limit.

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} \frac{KN_0}{N_0 + (K - N_0)e^{-rt}} = \frac{KN_0}{K} = K.$$

2(c). To find the concavity of the solution curve, one should consider the second derivative with respect to time. If the first derivative is give by equation (1). Then the second derivative is given by:

$$\frac{dN}{dt} = r\frac{dN}{dt}\left(1 - \frac{2N}{K}\right). \tag{2}$$

When equation (2) is positive the solution curve will be concave up. Equation (2) is positive when $\frac{dN}{dt} > 0$ and $1 - \frac{2N}{K} > 0$ or when $\frac{dN}{dt} < 0$ and $1 - \frac{2N}{K} < 0$. Careful consideration of the sign of $\frac{dN}{dt}$ shows that these conditions are satisfied when: $N < K$ and $N < K/2$ which simplifies to $N_0 < N < K/2$ or when $N > K$ which can only happen if $N_0 > K$.

2(d). When equation (2) is negative the solution curve will be concave down. Equation (2) is negative when $\frac{dN}{dt} > 0$ and $1 - \frac{2N}{K} < 0$ or when $\frac{dN}{dt} < 0$ and $1 - \frac{2N}{K} > 0$. Careful consideration of the sign of $\frac{dN}{dt}$ shows that these conditions are satisfied when: $N < K$ and $N > K/2$ which simplifies to $K/2 < N < K$ or when $N > K$ and $N < K/2$ which can never happen.

3(a). The model:

$$\frac{dN}{dt} = rKN^2 - rKMN - rN^3 + rMN^2, \quad r > 0, M < K. \tag{3}$$

The left-hand side can be re-written as $rN(M - N)(N - K)$; therefore $g(N) = r(M - N)(N - K)$. The graph is provided below:

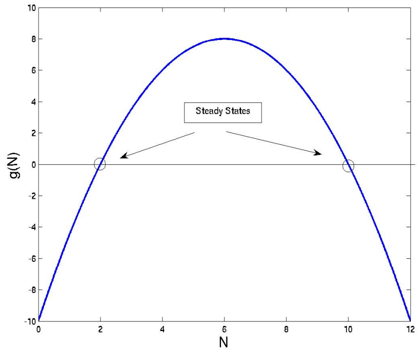

Figure 1: Plot of g(N) vs N for problem 3.

Notice that for this particular $g(N)$, the growth rate is negative for both large and small populations. The growth rate is only positive for a finite, intermediate range of population densities.

3(b). The steady states are N = 0, M, K. Stability is determined by letting $f(N) = rN(M - N)(N - K)$, $f'(N) = \frac{df}{dN} = 2rKN - rKM - 3rN^2 + 2rMN$. Evaluating the derivative at $N + 0$ results in $f'(0) = -rKM < 0$ and therefore N = 0 is always stable. Considering N = M and N = K in turn results in $f'(M) = rM(K - M) > 0$ and $f'(K) = -rK(K - M) < 0$. Therefore $N = M$ is always unstable and $N = K$ is always stable. The population will either approach K or 0. This means that there is a possibility of extinction if the initial population size is too small ($N_0 < M$). If the initial population size is sufficiently large, the population will approach its carrying capacity, K.

4(a). The Ricker model:

$$\frac{dN}{dt} = rNe^{-\beta N}. \tag{4}$$

The only steady state is $N = 0$ and since $f'(0) = r > 0$ it is always unstable. The growth function is $g(N) = re^{-\beta N}$ which decays *exponentially* to zero as N increases. Therefore the population will grow at an increasing slower rate, because $\lim_{N\to\infty} \frac{dN}{dt} = 0$.

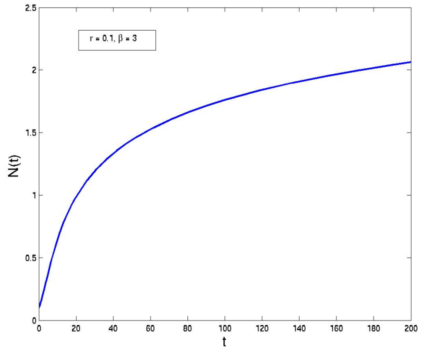

Figure 2: Plot of the solution of the Ricker model.

4(b). The Beverton-Holt model:

$$\frac{dN}{dt} = \frac{rN}{\alpha + N}. \tag{5}$$

The only steady state is $N = 0$ and since $f'(0) = \frac{r}{\alpha} > 0$ it is always unstable. The growth function is $g(N) = \frac{r}{\alpha+N}$ which decays *geometrically* to zero as N increases. In this case, $\lim_{N\to\infty} \frac{dN}{dt} = r$ which implies that the rate of change of the population will approach a constant. Therefore the population will eventually grow in a linear manner.

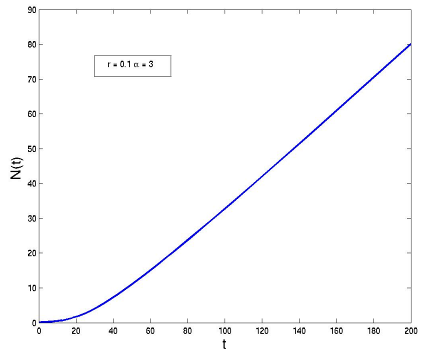

Figure 3: Plot of the solution of the Beverton-Holt model.