RESEARCH DESCRIPTION, AS OF SEPTEMBER 2008

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Residue calculus is a powerful tool in the study of many problems in algebra, geometry, and analysis, including effective versions of Hilbert's Nullstellensatz, [10], generalizations of the Jacobi vanishing theorem, [20], and explicit versions of the Ehrenpreis-Palamodov Fundamental Principle, [12]. Vaguely speaking the common idea in these applications is that residues can be seen as analytic representations of ideals of polynomials or holomorphic functions.

The classical multivariate residue theory concerns complete intersection ideals. With any complete intersection ideal $\mathfrak{a}$, that is, $\mathfrak{a}$ can be generated by $\operatorname{codim} \mathfrak{a}$ functions, one can associate in a canonical way (up to multiplication with a constant) a so-called residue current. The residue current represents $\mathfrak{a}$ in the sense that it has support on the variety of $\mathfrak{a}$ and the ideal of functions annihilating it is precisely $\mathfrak{a}$. If $\mathfrak{a}$ is generated by functions $f_1, \dots, f_r$, the residue current is given as the Coleff-Herrera product

$$R_{CH} = \bar{\partial}\left[\frac{1}{f_1}\right] \wedge \ldots \wedge \bar{\partial}\left[\frac{1}{f_r}\right],$$

introduced in [14]. The fact that $\operatorname{ann} R_{CH} = \{\varphi \text{ holomorphic }, \varphi R_{CH} = 0\} = \mathfrak{a}$ is referred to as the Duality Principle, since it generalizes the classical duality for Grothendieck residues, see [16]. It was proved independently by Passare [18] and Dickenstein-Sessa [15] and it has turned out to be useful in applications.

My research has been focused on the not necessarily complete intersection case. Basically I have worked in three different directions, which I briefly will discuss below. First I have worked on the general problem of finding a good notion of residue currents in the non-complete intersection case, resulting in the paper [7], see Section 3. Secondly I have studied a specific construction of residue currents - residue currents of Bochner-Martinelli type, introduced in [19] - see Sections 1 and 2. Thirdly, I have been interested in developing a theory for residue currents in general; in particular I have investigated how various properties of ideals are reflected in the corresponding residue currents, see Section 4.

1. Residue currents of Bochner-Martinelli type

The key step in the search for a good notion of residue currents in the not necessarily complete intersection case has turned out to be the paper [19], by

Passare, Tsikh and Yger, in which residue currents of Bochner-Martinelli type were introduced.

Let $f_1, \dots, f_r$ be a tuple of holomorphic functions. Then for each ordered index set $\mathcal{I} = \{i_1, \dots, i_k\} \subset \{1, \dots, r\}$ let $R^f_{\mathcal{I}}$ be the analytic continuation to $\lambda = 0$ of

$$\bar{\partial}|f|^{2\lambda} \wedge \sum_{\ell=1}^{k} (-1)^{\ell-1} \frac{\overline{f_{i_\ell}} \bigwedge_{q\neq\ell} \overline{df_{i_q}}}{|f|^{2k}}, \tag{1.1}$$

where $|f|^2 = |f_1|^2 + \ldots + |f_r|^2$. Then $R^f_{\mathcal{I}}$ is a well-defined $(0,k)$-current with support on $V(f) = \{f_1 = \ldots = f_r = 0\}$, that vanishes whenever $k < \operatorname{codim} V(f)$ or $k > \min(r,n)$. In case f defines a complete intersection, that is, $\operatorname{codim} V(f) = r$, the only non-vanishing current $R^f_{\{1,\dots,r\}}$ coincides with the Coleff-Herrera product R_{CH} of f.

Moreover $R^f_{\mathcal{I}}$ is annihilated by the integral closure of $\mathfrak{a}^k$. Recall that $\varphi \in \mathcal{O}_x$ belongs to the *integral closure* of $\mathfrak{a}$, denoted by $\overline{\mathfrak{a}}$, if $|\varphi| \leq C|f|$ for some constant C in some neighborhood of x or equivalently if φ satisfies a monic equation $\varphi^s + g_1\varphi^{s-1} + \ldots + g_s = 0$ with $g_i \in \mathfrak{a}^i$ for $1 \leq i \leq s$. Letting $\operatorname{ann} R^f$ denote the ideal $\{\varphi \text{ holomorphic}, \varphi R^f_{\mathcal{I}} = 0 \forall \mathcal{I}\}$ we have

$$\overline{\mathfrak{a}^\mu} \subseteq \operatorname{ann} R^f, \tag{1.2}$$

where μ denotes $\min(r,n)$.

In [1] the currents $R^f_{\mathcal{I}}$ were recovered as the coefficients of a residue current R^f constructed from the Koszul complex of f. From this construction follows that whenever a holomorphic function φ annihilates $R^f_{\mathcal{I}}$ for all $\mathcal{I}$, then $\varphi \in \mathfrak{a}$; in other words

$$\operatorname{ann} R^f \subseteq \mathfrak{a}, \tag{1.3}$$

so R^f satisfies one direction of the Duality Principle. If f is a complete intersection we have equality in (1.3) since $R^f_{\{1,\dots,r\}} = R_{CH}$, but in general the inclusion (1.3) is strict, compare to Section 2. Still $\operatorname{ann} R^f$ is big enough to in some sense captures the “size” of $\mathfrak{a}$. Combined with (1.2) it yields a new proof of the classical Briançon-Skoda theorem, [13]: $\overline{\mathfrak{a}^\mu} \subseteq \mathfrak{a}$, see also [3] and [6].

Residue currents of Bochner-Martinelli type have been used for various purposes, in particular for investigations in the non-complete intersection case, [11]. Vidras and Yger, [20], used residue currents of Bochner-Martinelli type to generalizations of Jacobi's theorem on vanishing of residues. In [2] and [4] the construction from [1] was used to obtain solutions to membership problems with control of the polynomial degree.

2. Residue currents of monomial ideals

In [21] I computed the current R^f in the case when $\mathfrak{a} = (f)$ is a monomial ideal. The main motivation was to investigate how far the Bochner-Martinelli currents are from giving a duality principle, and by that throw light on Briançon-Skoda type theorems. More specifically I wanted to investigate when the inclusion (1.3) is strict.

The theory of monomial ideals is one of the strong links between commutative algebra, algebraic geometry and combinatorics and it has been extensively developed in recent years, see for example [17]. Because of their simplicity and nice combinatorial description monomial ideals serve as a good toy model for illustrating general ideas and results in commutative algebra and algebraic geometry. Also many results can be proved by specializing to a monomial case. In fact, the existence of residue currents is proved by reducing to a monomial case by resolution of singularities. Monomial ideals are therefore a natural starting point for computing residue currents.

The main result in [21] is a combinatorial description of the annihilator ideal of R^f associated with a zero-dimensional monomial ideal $\mathfrak{a}$. In particular it follows that the inclusion (1.3) is always strict unless $\mathfrak{a}$ is a complete intersection. The proof amounts to computing currents in certain toric varieties.

3. Residue currents with prescribed annihilator ideals

In [7], which is a joint work with Mats Andersson, we extended the construction of residue currents of Bochner-Martinelli type from the Koszul complex in [1]. The aim was, given an ideal $\mathfrak{a}$ of germs of holomorphic functions, or more generally an ideal sheaf, to construct a residue current whose annihilator ideal is precisely $\mathfrak{a}$.

Consider an arbitrary complex of Hermitian holomorphic vector bundles over some complex manifold X

$$0 \to E_N \xrightarrow{F_N} \dots \xrightarrow{F_2} E_1 \xrightarrow{F_1} E_0, \tag{3.1}$$

that is exact outside an analytic variety Z of positive codimension. With this complex we associate a residue current R taking values in $\text{End}E$, where $E = \bigoplus E_j$, and with support on Z. This current measures in a certain way the lack of exactness of the associated complex of locally free sheaves of $\mathcal{O}$-modules

$$0 \to \mathcal{O}(E_N) \xrightarrow{F_N} \dots \xrightarrow{F_2} \mathcal{O}(E_1) \xrightarrow{F_1} \mathcal{O}(E_0). \tag{3.2}$$

Suppose that E_0 is of rank 1 and that (3.2) is exact; in other words, (3.2) is a free resolution of the ideal sheaf $\mathcal{J} = \text{Im}\,(\mathcal{O}(E_1) \to \mathcal{O}(E_0))$. The main result in [7] asserts that $\text{ann}\, R = \{\varphi \in \mathcal{O}(E_0);\ R\varphi = 0\}$ is in fact equal to $\mathcal{J}$. This characterization of an ideal (sheaf) as an annihilator of a residue current generalizes the Duality Principle for Coleff-Herrera currents. We used it to extend several results previously known for complete intersections, for example, we obtain a generalization Berndtsson-Passare's residue version of the Ehrenpreis-Palamodov Fundamental Principle, [12]. Also, these currents have recently been used by Andersson and Samuelsson to obtain new results for $\bar{\partial}$-equations on singular varieties, [5], and by Andersson, Samuelsson and Sznajdman, [6], to obtain versions of the Briançon-Skoda theorem on singular varieties. If (3.1) is the Koszul complex of a complete intersection $f_1, \ldots, f_r$ we get back the current $R^f = R_{CH}$ from [1].

The degree of explicitness of the current R depends of course directly on the degree of explicitness of (3.1). In simple cases such as a complete intersection, a resolution can be obtained from a (minimal) set of generators, but in general it is hard to find explicit resolutions. However, recently there has been a lot of work done for monomial ideals. In [22] I computed residue currents associated with cellular resolutions of monomial ideals, which were introduced in [9]. The main result in [22] is a complete description of the current in case $\mathfrak{a}$ is a so-called generic monomial ideal.

4. Decomposition of residue currents

Regarding residue currents as representations of ideals of holomorphic functions it is natural to ask how various properties of an ideal is reflected in the corresponding residue current. In [8], which is a joint work with Mats Andersson, and [22] we investigate how the residue currents from [7] admit natural decompositions corresponding to primary decomposition of ideals and irreducible decomposition of monomial ideals, respectively.

Recall that an ideal $\mathfrak{a}$ in a Noetherian ring A can be written as a finite intersection $\mathfrak{a} = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ of simpler, so called primary, ideals $\mathfrak{q}_i$; such a decomposition is called a primary decomposition of $\mathfrak{a}$. If $A = \mathbb{Z}$ the primary ideals are the ideals generated by prime powers and the primary decomposition of ideals just corresponds to a prime factorization of integers. If $\mathfrak{a} \subset \mathcal{O}_0$ is a complete intersection the components in a minimal primary decomposition corresponds to the irreducible components of the variety $V(\mathfrak{a})$.

Let R be a residue current constructed from a free resolution of an ideal $\mathfrak{a} \subset \mathcal{O}_0$ as in the previous section. In [8], which is a joint work with Mats Andersson, we show that R can be written $R = \sum_{\mathfrak{p}_i \in \mathrm{A}ss\mathfrak{a}} R^{\mathfrak{p}_i}$, where the sum is taken over the set of associated primes of $\mathfrak{a}$, that is, the set of ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}}_i$ in a minimal primary decomposition; here each current $R^{\mathfrak{p}_i}$ has support on $V(\mathfrak{p}_i)$ and $\operatorname{ann} R^{\mathfrak{p}_i}$ is primary. Moreover $\operatorname{ann} R = \bigcap_{\mathfrak{p}_i \in \mathrm{A}ss\mathfrak{a}} \operatorname{ann} R^{\mathfrak{p}_i}$ is a minimal primary decomposition of $\mathfrak{a}$.

As a tool we introduce a class of currents that we call pseudomeromorphic and that includes usual principal value and residue currents. The definition is modeled on the residue currents that appears in various works as [1] and [19]; a current is pseudomeromorphic if it can be written as a locally finite sum of push-forwards under holomorphic modifications of currents of the form

$$\left[\frac{1}{\sigma_{q+1}^{a_{q+1}}\cdots\sigma_n^{a_n}}\right]\bar\partial\left[\frac{1}{\sigma_1^{a_1}}\right]\wedge\cdots\wedge\bar\partial\left[\frac{1}{\sigma_q^{a_q}}\right]\wedge\alpha,$$

where σ_j are some local coordinates and α is a smooth form. In particular, all currents that appear in this text are pseudomeromorphic. An important property of pseudomeromorphic currents is that they allow for multiplication with characteristic functions of varieties. In fact, the current $R^{\mathfrak{p}_i}$ is defined as $R\mathbf{1}_{V(\mathfrak{p}_i)\setminus\bigcup_{\mathfrak{q}\in \mathrm{A}ss\mathfrak{a},q\supset p_i}V(\mathfrak{q})}$.

A monomial ideal can be written as a finite intersection of so-called irreducible monomial ideals; a monomial ideal is said to be irreducible if it is of the form $(z_1^{b_1},\dots,z_n^{b_n})$. Note that the irreducible decomposition is a refinement of the primary decomposition. In [22] it is shown that the residue current constructed from a resolution of a monomial ideal can be decomposed with respect to its irreducible decomposition. In fact, if the monomial ideal is zero-dimensional (and thereby primary), the components in the irreducible decomposition correspond the set of non-zero entries in the vector valued current R.

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