Math 676, completion of proof of Ostrowski’s theorem

**Theorem** (Ostrowski). Every archimedean absolute value on \( \mathbb{Q} \) is equivalent to the usual absolute value \(|\cdot|_\infty\), and every nontrivial non-archimedean absolute value on \( \mathbb{Q} \) is equivalent to the \( p \)-adic absolute value \(|\cdot|_p\) for some prime \( p \).

**Proof.** In class I proved the non-archimedean part of this. So let \(|\cdot|\) be an archimedean absolute value on \( \mathbb{Q} \). By the lemma from class, the archimedean hypothesis implies that there exist integers \( n > 1 \) for which \(|n|\) is arbitrarily large, so in particular there are infinitely many integers \( n > 1 \) for which \(|n| > 1\). Pick any integers \( m, n > 1 \) such that \(|n|, |m| > 1\). Write \( m = a_0 + a_1 n + \cdots + a_k n^k \) with \( a_i \in \{0, 1, \ldots, n-1\} \) and \( a_k \neq 0 \). Then \( m \geq n^k \), so that \( k \leq \log_n m \). Also the triangle inequality implies that, since \( k \leq \log_n m \), the archimedean absolute value on \( \mathbb{Q} \) is equivalent to the \( p \)-adic absolute value for some prime \( p \).

For any positive integer \( t \), we have \(|m|^t = |m| > 1\), so the above argument applies when we substitute \( m^t \) for \( m \), and yields

\[ |m|^t \leq (1 + t \log_n m) \cdot n \cdot |n|^{t \log_n m}. \]

Taking \( t \)-th roots gives

\[ |m| \leq \sqrt[1/t]{1 + t \log_n m} \cdot n^{1/t} \cdot |n|^{\log_n m}, \]

and since \( \lim_{t \to \infty} \left( \sqrt[1/t]{1 + t \log_n m} \cdot n^{1/t} \right) = 1 \) it follows that \(|m| \leq |n|^{\log_n m}\), or equivalently \( \log|m| \leq (\log_n m) \log|n| \), whence \( (\log|m|)/\log m \leq (\log|n|)/\log n \). The above argument applies if we interchange \( m \) and \( n \), and yields the reverse inequality, so we conclude that

\[ \frac{\log|m|}{\log m} = \frac{\log|n|}{\log n} \]

for all positive integers \( m, n \) such that \(|m|, |n| > 1\). Letting \( s \) be this common value, it follows that \(|m| = m^s\) for all positive integers \( m \) such that \(|m| > 1\). Note that \( s > 0 \). Next, for any positive integer \( \ell \) we have \(|\ell| > 0\), so there is some positive integer \( m \) for which both \(|m\ell|\) and \(|m|\) are larger than 1, whence

\[ |\ell| = \frac{|m\ell|}{|m|} = \frac{(m\ell)^s}{m^s} = \ell^s. \]

Thus \(|\ell| = \ell^s\) for all positive integers \( \ell \), and hence (by multiplicativity of absolute values) for all positive rational numbers \( \ell \). Finally, since \(|-1| = 1\) and \(|0| = 0\), it follows that \(|\ell| = |\ell|^s_\infty\) for all \( \ell \in \mathbb{Q} \), so that \(|\cdot|\) is equivalent to \(|\cdot|_\infty\). \( \square \)