PERMUTATION POLYNOMIALS ON $\mathbb{F}_q$ INDUCED FROM RÉDEI FUNCTION BIJECTIONS ON SUBGROUPS OF $\mathbb{F}_q^*$

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Abstract. We construct classes of permutation polynomials over $\mathbb{F}_{Q^2}$ by exhibiting classes of low-degree rational functions over $\mathbb{F}_{Q^2}$ which induce bijections on the set of $(Q + 1)$-th roots of unity. As a consequence, we prove two conjectures about permutation trinomials from a recent paper by Tu, Zeng, Hu and Li.

1. Introduction

A polynomial $f(x) \in \mathbb{F}_q[x]$ is called a permutation polynomial if the function $\alpha \mapsto f(\alpha)$ induces a permutation of $\mathbb{F}_q$. Since they were first studied in the mid-19th century, one of the driving questions about permutation polynomials has been to construct examples having especially simple shapes. This actually requires polynomials which are nice in two ways: they have a simple algebraic form, and also they induce a function on $\mathbb{F}_q$, which has the nice combinatorial property of being a permutation. The vast majority of known examples of “nice” permutation polynomials have the form $x^r h(x^d)$ where $h \in \mathbb{F}_q[x]$ and $d > 1$ is a divisor of $q - 1$. The reason this form is special is that a general result (see Lemma 2.2) asserts that $x^r h(x^d)$ permutes $\mathbb{F}_q$ if and only if $\gcd(r, d) = 1$ and $x^r h(x^d)$ permutes the set of $(q-1)/d$-th roots of unity in $\mathbb{F}_q^*$. This leads to the question of producing collections of polynomials which permute the set $\mu_k$ of $k$-th roots of unity in $\mathbb{F}_q$ for certain values of $k$. There are two simple types of polynomials which permute $\mu_k$: for arbitrary $k$ one can use polynomials of the form $\beta x^n + (x^k - 1) \cdot g(x)$ where $\beta \in \mu_k$ and $\gcd(n, k) = 1$, and if $k = Q - 1$ where $q = Q^r$ then one can use $h(x) - h(0)$ where $h(x) \in \mathbb{F}_Q[x]$ permutes $\mathbb{F}_Q$. These two simple types of permutations of $\mu_k$ account for essentially all known permutation polynomials over finite fields. Indeed, it is difficult to identify any other polynomials having “nice” form which permute $\mu_k$. In this paper we present classes of permutation polynomials obtained by a new variant of this construction, in which

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• $\mu_k$ is not the multiplicative group of a subfield of $\mathbb{F}_q$
• the induced function on $\mu_k$ is most naturally presented as a rational function rather than a polynomial.

It is perhaps surprising that there are permutations of $\mu_k$ which can be represented by a rational function having an especially simple form, but which cannot be represented by an especially simple polynomial. We obtain the following classes of permutation polynomials.

**Theorem 1.1.** Let $Q$ be a prime power, let $n > 0$ and $k \geq 0$ be integers, and let $\beta, \gamma \in \mathbb{F}_{Q^2}$ satisfy $\beta^{Q+1} = 1$ and $\gamma^{Q+1} \neq 1$. Then

$$f(x) := x^{n+k(Q+1)} \cdot \left( (\gamma x^{Q-1} - \beta)^n - \gamma(x^{Q-1} - \gamma^Q \beta)^n \right)$$

permutes $\mathbb{F}_{Q^2}$ if and only if $\gcd(n+2k, Q-1) = 1$ and $\gcd(n, Q+1) = 1$.

**Theorem 1.2.** Let $Q$ be a prime power, let $n, k$ be integers with $n > 0$ and $k \geq 0$, and let $\beta, \delta \in \mathbb{F}_{Q^2}$ satisfy $\beta^{Q+1} = 1$ and $\delta \notin \mathbb{F}_Q$. Then

$$f(x) := x^{n+k(Q+1)} \cdot \left( (\delta x^{Q-1} - \beta \delta^Q)^n - \delta(x^{Q-1} - \beta)^n \right)$$

permutes $\mathbb{F}_{Q^2}$ if and only if $\gcd(n(n+2k), Q-1) = 1$.

The following corollary illustrates these results in the special case $n = 3$, for certain values of $\beta, \gamma, \delta$.

**Corollary 1.3.** Let $Q$ be a prime power, and let $k$ be a nonnegative integer. The polynomial $g(x) := x^k(Q+1)+3 + 3x^k(Q+1)+Q + x^k(Q+1)+3Q$ permutes $\mathbb{F}_{Q^2}$ if and only if $\gcd(2k+3, Q-1) = 1$ and $3 \nmid Q$.

Specializing even further to the values $k = Q - 3$, $k = 1$, and $k = 0$ yields the following consequence.

**Corollary 1.4.** Let $Q$ be a prime power with $3 \nmid Q$. Then

1. $x^Q + 3x^{2Q-1} - x^{Q^2+1}$ is a permutation polynomial over $\mathbb{F}_{Q^2}$.
2. $x^{Q^4+4} + 3x^{2Q+3} - x^{4Q+1}$ is a permutation polynomial over $\mathbb{F}_{Q^2}$ if and only if $Q \equiv 1 \pmod{5}$.
3. $x^3 + 3x^{Q^2+2} - x^{3Q}$ is a permutation polynomial over $\mathbb{F}_{Q^2}$ if and only if $Q \equiv 2 \pmod{3}$.

In case $Q = 2^{2m+1}$, the first two parts of this corollary were conjectured by Tu, Zeng, Hu and Li [7]. Conversely, these conjectures were the impetus which led to the present paper.

The proofs of our results rely on exhibiting certain permutations of the set of $(Q+1)$-th roots of unity in $\mathbb{F}_{Q^2}$. The permutations we exhibit are represented by Rédei functions, namely, rational functions over a field $K$ which have the form $\mu \circ x^n \circ \mu^{-1}$ where $\mu(x)$ is a degree-one rational function having coefficients in an extension of $K$, and $\mu^{-1}$ is the
rational function such that \( \mu^{-1}(\mu(x)) = x \). For further results about such functions, see for instance [2, 5, 6] and [3, Ch. 5]. Although permutation polynomials on subgroups of \( F_q^* \) have also been studied [1], the present paper is the first to examine Rédei functions as permutations of such subgroups, and especially the first to notice that Rédei functions can permute subgroups of \( F_q^* \) other than the multiplicative groups of subfields of \( F_q \).

We prove Theorems 1.1 and 1.2 in the next two sections, and deduce the corollaries in Section 4. We conclude this paper by using our approach to give a very simple proof of a substantial generalization of the main result of [7].

2. Proof of Theorem 1.1

In this section we prove Theorem 1.1. Throughout this section, \( Q \) is a prime power, and we write \( \mu_d \) for the set of \( d \)-th roots of unity in the algebraic closure \( \overline{F}_Q \) of \( F_Q \). We begin with a lemma exhibiting the permutations of \( \mu_{Q+1} \) induced by degree-one rational functions.

**Lemma 2.1.** Let \( Q \) be a prime power, and let \( \ell(x) \in \overline{F}_Q(x) \) be a degree-one rational function. Then \( \ell \) induces a bijection on \( \mu_{Q+1} \) if and only if \( \ell(x) \) equals either

- \( \beta/x \) with \( \beta \in \mu_{Q+1} \), or
- \( (x - \gamma^Q \beta)/(\gamma x - \beta) \) with \( \beta \in \mu_{Q+1} \) and \( \gamma \in \overline{F}_Q^2 \setminus \mu_{Q+1} \).

**Proof.** If \( \ell(x) \) induces a bijection on \( \mu_{Q+1} \), then it maps at least \( Q+1 \geq 3 \) elements of \( \overline{F}_Q^2 \) into \( \overline{F}_Q^2 \); since a degree-one rational function can be defined over any field which contains three points and their images, it follows that \( \ell(x) \) is in \( \overline{F}_Q^2(x) \). Thus, we may assume that \( \ell(x) = (\alpha x - \delta)/(\gamma x - \beta) \) with \( \alpha, \beta, \gamma, \delta \in \overline{F}_Q^2 \) and \( \alpha \beta \neq \gamma \delta \). If \( \ell(x) \) permutes \( \mu_{Q+1} \) then in particular \( \mu_{Q+1} \) does not contain \( \ell^{-1}(\infty) = \beta/\gamma \).

Henceforth assume that \( \mu_{Q+1} \) does not contain \( \beta/\gamma \). Then \( \ell(x) \) induces a bijection on \( \mu_{Q+1} \) if and only if the numerator of \( \ell(x)^{Q+1} - 1 \) is divisible by \( x^{Q+1} - 1 \). This numerator is \( (\alpha x - \delta)^{Q+1} - (\gamma x - \beta)^{Q+1} \), which equals

\[
(\alpha^{Q+1} - \gamma^{Q+1})x^{Q+1} - (\alpha^{Q} \delta - \gamma^{Q} \beta)x^Q - (\alpha \delta^Q - \gamma \beta^Q)x + \delta^{Q+1} - \beta^{Q+1}.
\]

This polynomial is divisible by \( x^{Q+1} - 1 \) if and only if it equals \( (\alpha^{Q+1} - \gamma^{Q+1})(x^{Q+1} - 1) \), or equivalently

\[
\alpha^{Q+1} = \gamma^{Q+1} \quad \text{and} \quad \alpha^{Q+1} + \delta^{Q+1} = \gamma^{Q+1} + \beta^{Q+1}.
\]

If \( \alpha = 0 \) then, since \( \alpha^{Q} \delta = \gamma^{Q} \beta \) but \( \alpha \beta \neq \gamma \delta \), we must have \( \beta = 0 \), and then \( \delta^{Q+1} = \gamma^{Q+1} \), whence \( f(x) = -\delta/(\gamma x) \) where \( (-\delta/\gamma)^{Q+1} = 1 \). Now suppose \( \alpha \neq 0 \), so that upon dividing \( \alpha, \delta, \gamma, \beta \) by \( \alpha \) we may
assume that $\alpha = 1$. Then $\delta = \gamma Q \beta$ and $\gamma Q + 1 + \beta Q + 1 = \alpha Q + 1 + \delta Q + 1 = 1 + \gamma Q + 1 + \beta Q + 1$, so that $(\gamma Q + 1 - 1)(\beta Q + 1 - 1) = 0$ and thus either $\gamma Q + 1 = 1$ or $\beta Q + 1 = 1$. But $0 \neq \alpha \beta - \gamma \delta = \beta (1 - \gamma Q + 1)$ implies that $\gamma Q + 1 \neq 1$, so $\beta Q + 1 = 1$. Finally, in this case the condition $\beta / \gamma \notin \mu_{Q + 1}$ asserts that $\gamma \notin \mu_{Q + 1}$.

The next lemma was first proved in [8].

**Lemma 2.2.** Pick $h \in \mathbb{F}_q[x]$ and integers $d, r > 0$ such that $d \mid (q - 1)$. Then $f(x) := x^r h(x^{(q-1)/d})$ permutes $\mathbb{F}_q$ if and only if both

1. $\gcd(r, (q - 1)/d) = 1$ and
2. $x^r h(x)^{(q-1)/d}$ permutes $\mu_d$.

This lemma has been used in several investigations of permutation polynomials, for instance see [4, 8, 9, 10, 11]. Since the proof of Lemma 2.2 is short, we include it here for the reader’s convenience.

**Proof.** Write $s := (q - 1)/d$. For $\zeta \in \mu_s$, we have $f(\zeta x) = \zeta^r f(x)$. Thus, if $f$ permutes $\mathbb{F}_q$ then $\gcd(r, s) = 1$. Conversely, if $\gcd(r, s) = 1$ then the values of $f$ on $\mathbb{F}_q$ consist of all the $s$-th roots of the values of $f(x)^s = x^{rs} h(x)^s$.

But the values of $f(x)^s$ on $\mathbb{F}_q$ consist of $f(0)^s = 0$ and the values of $g(x) := x^r h(x)^s$ on $(\mathbb{F}_q^*)^s$. Thus, $f$ permutes $\mathbb{F}_q$ if and only if $g$ permutes $(\mathbb{F}_q^*)^s = \mu_d$. \qed

We now prove Theorem 1.1.

**Proof of Theorem 1.1.** Write $h(x) := (\gamma x - \beta)^n - (\gamma (x - \gamma Q \beta))^n$ and $r := n + k(Q + 1)$. By Lemma 2.2, $f(x) = x^r h(x^{Q - 1})$ permutes $\mathbb{F}_q$ if and only if $\gcd(n + k(Q + 1), Q - 1) = 1$ and $g(x) := x^r h(x)^{Q - 1}$ permutes $\mu_{Q + 1}$. Henceforth we assume that $\gcd(n + k(Q + 1), Q - 1) = 1$, or equivalently $\gcd(n + 2k, Q - 1) = 1$; note that this implies $n$ is odd if $Q$ is odd, so that $(-1)^n = -1$ in $\mathbb{F}_Q$.

We begin by showing that $h(x)$ has no roots in $\mu_{Q + 1}$. For $\alpha \in \mu_{Q + 1}$, if $h(\alpha) = 0$ then $\delta := (\gamma \alpha - \beta)/(\alpha - \gamma Q \beta)$ satisfies $\delta^n = \gamma$, so in particular $\delta \notin \mu_{Q + 1}$. But we compute

$$\delta^Q = \frac{\gamma Q \alpha^{-1} - \beta^{-1}}{\alpha^{-1} - \gamma \beta^{-1}} = \frac{\gamma Q \beta - \alpha}{\beta - \gamma \alpha} = \delta^{-1},$$

which is impossible since $\delta \notin \mu_{Q + 1}$. Hence $h(x)$ has no roots in $\mu_{Q + 1}$, so $h(\mu_{Q + 1}) \subseteq \mathbb{F}_q^*$, whence $g(\mu_{Q + 1}) \subseteq \mu_{Q + 1}$. Thus, $g$ permutes $\mu_{Q + 1}$ if and only if $g$ is injective on $\mu_{Q + 1}$.
Next, for \( \alpha \in \mu_{Q+1} \) we compute
\[
h(\alpha)^Q = (\gamma^Q \alpha^{-1} - \beta^{-1})^n - \gamma^Q (\alpha^{-1} - \beta^{-1})^n = \frac{(\gamma^Q \beta - \alpha)^n - \gamma^Q (\beta - \gamma \alpha)^n}{(\beta \alpha)^n},
\]
so that
\[
g(\alpha) = \alpha^r - n \beta^{-n} \frac{(\gamma^Q \beta - \alpha)^n - \gamma^Q (\beta - \gamma \alpha)^n}{(\gamma \alpha - \beta)^n - \gamma (\alpha - \gamma Q \beta)^n}.
\]
Since \( r - n = k(Q + 1) \) and \( \alpha^{Q+1} = 1 \), it follows that \( g \) is injective on \( \mu_{Q+1} \) if and only if
\[
G(x) := \beta \frac{(\gamma^Q \beta - x)^n - \gamma^Q (\beta - \gamma x)^n}{(\gamma x - \beta)^n - \gamma (x - \gamma Q \beta)^n}
\]
is injective on \( \mu_{Q+1} \). For \( \ell(x) := (x - \gamma Q \beta)/(\gamma x - \beta) \), we have
\[
G = \ell^{-1} \circ x^n \circ \ell,
\]
so \( G \) is injective on \( \mu_{Q+1} \) if and only if \( x^n \) is injective on \( \ell(\mu_{Q+1}) \). By Lemma 2.1 we have \( \ell(\mu_{Q+1}) = \mu_{Q+1} \), so \( x^n \) is injective on this set if and only if \( \gcd(n, Q + 1) = 1 \). This concludes the proof. \( \square \)

3. Proof of Theorem 1.2

In this section we prove Theorem 1.2. As in the previous section, \( Q \) is a prime power, and \( \mu_d \) denotes the set of \( d \)-th roots of unity in \( \mathbb{F}_q \). We begin by determining the bijections \( \mu_{Q+1} \rightarrow \mathbb{F}_q \cup \{\infty\} \) which are induced by degree-one rational functions.

**Lemma 3.1.** Let \( Q \) be a prime power, and let \( \ell \in \mathbb{F}_q(x) \) be a degree-one rational function. Then \( \ell(x) \) induces a bijection from \( \mu_{Q+1} \) to \( \mathbb{F}_q \cup \{\infty\} \) if and only if \( \ell(x) = (\delta x - \gamma)/(\alpha x - \beta) \) with \( \beta \in \mu_{Q+1} \) and \( \delta \in \mathbb{F}_{Q^2} \setminus \mathbb{F}_q \).

**Proof.** If \( \ell \) maps \( \mu_{Q+1} \) into \( \mathbb{F}_q \cup \{\infty\} \), then \( \ell \) maps at least \( Q + 1 \geq 3 \) elements of \( \mathbb{F}_{Q^2} \) into \( \mathbb{F}_q \cup \{\infty\} \), so \( \ell \in \mathbb{F}_{Q^2}(x) \). Thus we may write \( \ell = (\delta x - \gamma)/(\alpha x - \beta) \) with \( \delta, \gamma, \alpha, \beta \in \mathbb{F}_{Q^2} \) and \( \delta \neq \gamma \). Moreover, we may assume that \( \ell^{-1}(\infty) = \beta/\alpha \) is in \( \mu_{Q+1} \), so that after suitably scaling \( \delta, \gamma, \alpha, \beta \) we may assume that \( \alpha = 1 \) and \( \beta \in \mu_{Q+1} \). Under these hypotheses, \( \ell \) induces a bijection from \( \mu_{Q+1} \) to \( \mathbb{F}_q \cup \{\infty\} \) if and only if the numerator of \( \ell(x)^Q - \ell(x) \) is divisible by \( (x^{Q^2+1} - 1)/(x - \beta) \). This numerator is
\[
(\delta^Q x^Q - \gamma^Q)(x - \beta) - (x^Q - \beta^Q)(\delta x - \gamma)
= (\delta^Q - \delta)x^{Q+1} + (\gamma - \beta \delta^Q)x^Q + (\delta \beta^Q - \gamma^Q)x + (\gamma^Q \beta - \gamma \beta^Q),
\]
which is congruent mod \( x^{Q+1} - 1 \) to
\[
g(x) := (\gamma - \beta \delta^Q)x^Q + (\delta \beta^Q - \gamma^Q)x + (\delta^Q - \delta + \gamma \beta^Q - \gamma \beta^Q).
\]
Since
\[ \frac{x^{Q+1} - 1}{x - \beta} = \frac{x^{Q+1} - \beta^{Q+1}}{x - \beta} = \sum_{i=0}^{Q} x^i \beta^{Q-i}, \]

it follows that if \( Q > 2 \) then \( g(x) \) is divisible by \((x^{Q+1} - 1)/(x - \beta)\) if and only if \( g(x) \) is the zero polynomial, or equivalently
\[ \gamma = \beta \delta^Q \quad \text{and} \quad \delta + \gamma \beta^Q \in \mathbb{F}_Q. \]

Since \( \beta \in \mu_{Q+1} \), the second condition follows from the first, as \( \delta + \gamma \beta^Q = \delta + \beta \delta^Q \beta^Q + 1 = \delta + \delta^Q \) is in \( \mathbb{F}_Q \). If these conditions hold then \( \delta \beta - \gamma = \beta(\delta - \delta^Q) \) is nonzero precisely when \( \delta \notin \mathbb{F}_Q \). Finally, one can easily check that the same conclusion holds when \( Q = 2 \). \( \square \)

We now prove Theorem 1.2.

Proof of Theorem 1.2. Write \( h(x) := (\delta x - \beta \delta^Q)^n - \delta(x - \beta)^n \) and \( r := n + k(Q + 1) \). By Lemma 2.2, \( f(x) = x^r h(x^{Q-1}) \) permutes \( \mathbb{F}_Q^* \) if and only if \( \gcd(n + k(Q + 1), Q - 1) = 1 \) and \( g(x) := x^r h(x^{Q-1}) \) permutes \( \mu_{Q+1} \). Henceforth we assume that \( \gcd(n + k(Q + 1), Q - 1) = 1 \), or equivalently \( \gcd(n + 2k, Q - 1) = 1 \); note that this implies \( n \) is odd if \( Q \) is odd, so that \((-1)^n = -1\) in \( \mathbb{F}_Q \).

We begin by showing that \( h(x) \) has no roots in \( \mu_{Q+1} \). Our hypothesis \( \delta \notin \mathbb{F}_Q \) implies that \( h(\beta) = (\delta \beta - \beta \delta^Q)^n = \beta^n (\delta - \delta^Q)^n \neq 0 \). For \( \alpha \in \mu_{Q+1} \setminus \{\beta\} \), if \( h(\alpha) = 0 \) then \( \theta := (\delta \alpha - \beta \delta^Q)/(\alpha - \beta) \) satisfies \( \theta^n = \delta \), so in particular \( \theta \notin \mathbb{F}_Q \). But we compute
\[ \theta^Q = \frac{\delta^Q \alpha^{-1} \beta^{-1} \delta}{\alpha^{-1} - \beta^{-1}} = \frac{\delta^Q \beta - \alpha \delta}{\beta - \alpha} = \theta, \]

which is a contradiction. Hence \( h(x) \) has no roots in \( \mu_{Q+1} \), so \( h(\mu_{Q+1}) \subseteq \mathbb{F}_Q^* \), whence \( g(\mu_{Q+1}) \subseteq \mu_{Q+1} \). Thus, \( g \) permutes \( \mu_{Q+1} \) if and only if \( g \) is injective on \( \mu_{Q+1} \).

Next, for \( \alpha \in \mu_{Q+1} \) we compute
\[ h(\alpha)^Q = (\delta^Q \alpha^{-1} - \beta^{-1} \delta)^n - \delta^Q (\alpha^{-1} - \beta^{-1})^n = \frac{(\delta^Q \beta - \alpha \delta)^n - \delta^Q (\beta - \alpha)^n}{(\alpha \beta)^n}, \]

so that
\[ g(\alpha) = \alpha^{r-n} \beta^{-n} \frac{(\delta^Q \beta - \alpha \delta)^n - \delta^Q (\beta - \alpha)^n}{(\delta \alpha - \beta \delta^Q)^n - \delta (\alpha - \beta)^n}. \]

Since \( r - n = k(Q + 1) \) and \( \alpha^{Q+1} = 1 \), it follows that \( g \) is injective on \( \mu_{Q+1} \) if and only if
\[ G(x) := -\beta \frac{(\delta^Q \beta - x \delta)^n - \delta^Q (\beta - x)^n}{(\delta x - \beta \delta^Q)^n - \delta (x - \beta)^n} \]
is injective on $\mu_{Q+1}$. For $\ell(x) := (\delta x - \beta \delta^Q)/(x - \beta)$, we have

$$G = \ell^{-1} \circ x^n \circ \ell.$$ 

so $G$ is injective on $\mu_{Q+1}$ if and only if $x^n$ is injective on $\ell(\mu_{Q+1})$. By Lemma 3.1 we have $\ell(\mu_{Q+1}) = \mathbb{F}_Q \cup \{\infty\}$, so $x^n$ is injective on this set if and only if $\gcd(n, Q - 1) = 1$. This concludes the proof. \hfill \Box

4. PROOFS OF COROLLARY 1.3 AND COROLLARY 1.4

In this section we prove Corollary 1.3 and Corollary 1.4.

Proof of Corollary 1.3. If $Q \equiv 0 \pmod{3}$ then $g(1) = 0 = g(0)$ so $g(x)$ does not permute $\mathbb{F}_{Q^2}$. If $Q \equiv 1 \pmod{3}$ then put $n = 3$ and $\beta = 1$, and let $\gamma$ be a primitive cube root of unity in $\mathbb{F}_{Q^2}$. In this case, Theorem 1.1 says that $(\gamma - 1)g(x)$ permutes $\mathbb{F}_{Q^2}$ if and only if $\gcd(3 + 2k, Q - 1) = 1$. Finally, if $Q \equiv 2 \pmod{3}$ then put $n = 3$ and $\beta = \delta$, where $\delta$ is a primitive cube root of unity in $\mathbb{F}_{Q^2}$. In this case, Theorem 1.2 says that $(\delta - 1)g(x)$ permutes $\mathbb{F}_{Q^2}$ if and only if $\gcd(3 + 2k, Q - 1) = 1$. \hfill \Box

Proof of Corollary 1.4. Items (2) and (3) follow at once from the cases $k = 1$ and $k = 0$ of Corollary 1.3. The case $k = Q - 3$ of Corollary 1.3 asserts that $g(x) := x^{Q^2 - 2Q} + 3x^{Q^2 - Q - 1} - x^{Q + 1}$ is a permutation polynomial over $\mathbb{F}_{Q^2}$ if and only if $\gcd(2Q - 3, Q - 1) = 1$, which always holds. Thus $g(x^{Q^2 - 2})$ is a permutation polynomial over $\mathbb{F}_{Q^2}$, as is the reduction of $g(x^{Q^2 - 2}) \mod x^{Q^2} - x$. Since this reduction equals $x^{2Q - 1} + 3x^Q - x^{Q^2 - Q + 1}$, item (1) of Corollary 1.4 follows. \hfill \Box

5. THE MAIN RESULT OF [7]

In this section we give a simple proof of a generalization of [7, Thm. 1]. Our proof is completely different from the one in [7]. Once again, $\mu_{Q+1}$ denotes the set of $(Q+1)$-th roots of unity in $\mathbb{F}_Q$.

**Theorem 5.1.** Let $Q$ be a prime power, let $r$ be a positive integer, and let $\beta$ be a $(Q+1)$-th root of unity in $\mathbb{F}_{Q^2}$. Let $h(x) \in \mathbb{F}_{Q^2}[x]$ be a polynomial of degree $d$ such that $h(0) \neq 0$ and

$$(x^d \cdot h(1/x))^Q = \beta \cdot h(x^Q).$$

Then $f(x) := x^r h(x^{Q-1})$ permutes $\mathbb{F}_{Q^2}$ if and only if all of the following hold:

1. $\gcd(r, Q - 1) = 1$
2. $\gcd(r - d, Q + 1) = 1$
3. $h(x)$ has no roots in $\mu_{Q+1}$.
Remark 5.2. The polynomials \( h(x) \) satisfying the hypotheses of Theorem 5.1 can be described explicitly in terms of their coefficients. They are \( h(x) = \sum_{i=0}^{d} a_i x^i \) where \( a_0 \neq 0 \) and, for \( 0 \leq i \leq d/2 \), we have \( a_i \in \mathbb{F}_{Q^2} \) and \( a_{d-i} = (\beta a_i)^Q \).

Proof of Theorem 5.1. By Lemma 2.2, we see that \( f(x) \) permutes \( \mathbb{F}_{Q^2} \) if and only if \( \gcd(r, Q-1) = 1 \) and \( g(x) := x^r h(x)^{Q-1} \) permutes \( \mu_{Q+1} \). We may assume that \( h(x) \) has no roots in \( \mu_{Q+1} \), since otherwise \( g \) cannot permute \( \mu_{Q+1} \). Then any \( \alpha \in \mu_{Q+1} \) satisfies

\[
g(\alpha) = \alpha^r \frac{h(\alpha)^Q}{h(\alpha)} = \alpha^r \frac{h(\alpha^{-Q})^Q}{h(\alpha)} = \alpha^{r-d} \beta,
\]

so \( g \) permutes \( \mu_{Q+1} \) if and only if \( \gcd(r-d, Q+1) = 1 \). \(\square\)

We now illustrate Theorem 5.1 in the special case \( h(x) = x^d + \beta^{-1} \).

Corollary 5.3. Let \( Q \) be a prime power, let \( r \) and \( d \) be positive integers, and let \( \beta \) be a \((Q+1)\)-th root of unity in \( \mathbb{F}_{Q^2} \). Then \( x^{r+d(Q-1)} + \beta^{-1} x^r \) permutes \( \mathbb{F}_{Q^2} \) if and only if all of the following hold:

\[
(1) \quad \gcd(r, Q-1) = 1
\]

\[
(2) \quad \gcd(r-d, Q+1) = 1
\]

\[
(3) \quad (-\beta)^{(Q+1)/\gcd(Q+1,d)} \neq 1.
\]

Proof. Since \( h(x) := x^d + \beta^{-1} \) satisfies the hypotheses of Theorem 5.1, the Corollary will follow from Theorem 5.1 once we show that the final conclusion in the Corollary is equivalent to the final conclusion in the Theorem. For this, note that \( h(x) \) has roots in \( \mu_{Q+1} \) if and only if \(-\beta^{-1}\) is in \( (\mu_{Q+1})^d \), which equals \( \mu_{(Q+1)/\gcd(Q+1,d)} \). \(\square\)

In case \( Q \) is even, Corollary 5.3 is a refinement of [7, Thm. 1].

References


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