1. Show that the series given below are convergent and in each case find the smallest value of $n$ which ensures that the $n$th partial sum $s_n$ is accurate to within $10^{-6}$.
   a) $\sum_{n=1}^{\infty} \frac{1}{n^4}$
   b) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$

2. Recall from hw9: $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t) dt$.
   a) Now show that $f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \frac{f^{(4)}(a)}{4!} \int_a^x (x-t)^4 f^{(4)}(t) dt$. (hint: in the result from hw9, set $u = f'''(t), dv = \frac{(x-t)^2}{2}dt$, and integrate by parts)
   b) Define the function $T_3(x)$ as below.
   
   
   $T_3(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3$

   Note that $T_3(x)$ is a cubic function of $x$; it is called the Taylor polynomial of degree 3 for $f(x)$ at $x = a$. Show that $T_3(x)$ and $f(x)$ have the same function value, 1st derivative value, 2nd derivative value, and 3rd derivative value at $x = a$. We view $T_3(x)$ as a cubic approximation to $f(x)$ near the point $x = a$.

   c) Note that part (a) says, $f(x) = T_3(x) + \int_a^x \frac{(x-t)^3}{3!} f^{(4)}(t) dt$.
   In this case the error is the difference between the given function $f(x)$ and the cubic approximation $T_3(x)$. Show that the error satisfies the bound $|f(x) - T_3(x)| \leq \frac{1}{4!} M_4 |x-a|^4$, where $M_4 = \max |f^{(4)}(t)|$. This implies that if $x$ is close to $a$, then $T_3(x)$ is a very very good approximation to $f(x)$.

   d) In each case below find $T_3(x)$ and sketch $f(x), T_1(x), T_2(x), T_3(x)$ on the same graph.
   
   (i) $f(x) = e^x, \ a = 0$   (ii) $f(x) = \sin x, \ a = 0$   (iii) $f(x) = \sin x, \ a = \frac{\pi}{4}$