Homework #2. To be handed in on Monday, October 3.

1. Let $u$ be a locally bounded function on a set $E \subset \mathbb{R}^n$ that takes values in $[-\infty, +\infty)$. The \textit{upper semicontinuous regularization} of $u$ is the function $u^*$ defined by

$$u^*(x) = \lim_{y \to x, y \in E} \sup u(y).$$

Prove that $u^*$ is upper semicontinuous on $E$.

2. Recall that a set $\Omega \subset \mathbb{R}^n$ is convex if the line segment from $x$ to $y$, $[x, y] := \{(1-t)x + ty : 0 \leq t \leq 1\} \subset \Omega$ whenever $x, y \in \Omega$. A function $\varphi$ is said to be convex on $\Omega$ if and only if its graph lies below any chord; i.e.

$$\varphi((1-t)x + ty) \leq (1-t)\varphi(x) + t\varphi(y)$$

for $x, y \in \Omega$ and $0 \leq t \leq 1$.

(a) Prove that a convex function $\varphi$ on an open convex set $\Omega$ is Lipschitz continuous of order 1 on each compact subset of $\Omega$; that is, for each compact set $K \subset \Omega$, there is a constant $M$ such that

$$|\varphi(x) - \varphi(y)| \leq M|x - y|$$

for all $x, y \in K$. Hint: Look first at the one variable case and find an explicit bound for the difference quotients of $\varphi$.

(b) Prove that a $C^2$ function $\varphi$ on an open set is convex in a neighborhood of each point of the set if and only if the Hessian of $\varphi$, $\left[ \frac{\partial^2 \varphi}{\partial x_j \partial x_k} \right]$, is a positive (semi)definite matrix. That is, for all $\lambda \in \mathbb{R}^n$ and $x \in \Omega$,

$$\sum_{j,k=1}^{n} \lambda_j \lambda_k \frac{\partial^2 \varphi(x)}{\partial x_j \partial x_k} \geq 0.$$ 

3. (a) Show that a function $u$ that is independent of $\text{Im} z$ is plurisubharmonic if and only if it is a convex function of $\text{Re} z$ that is nondecreasing in each variable. In case $u$ is smooth, what is the relationship between the real Hessian of the convex function and the complex Hessian $\left[ \frac{\partial^2 u}{\partial z_j \partial z_k} \right]$ of $u$?

(b) Show that a function $u$ that is a function of $(|z_1|, \ldots, |z_n|)$ is plurisubharmonic if and only if it is a convex function of $(\log |z_1|, \ldots, \log |z_n|)$ that is increasing in each variable separately. That is, there is a convex function $\varphi$ that is nondecreasing in each variable and satisfies...
\( u(z) = \varphi(\log |z_1|, \ldots, \log |z_n|) \). In case \( u \) is smooth, give the relationship between the complex Hessian of \( u \) and that of \( \varphi \) in the smooth case.

4. Let \( p_\nu(z) \) be a sequence of homogeneous polynomials in \( z = (z_1, \ldots, z_n) \) with \( p_\nu \) of degree \( \nu \), \( \nu = 0, 1, 2, \ldots \). Suppose also that: such that a formal power series of . Suppose that:

(a) the power series \( \sum_{\nu=0}^\infty p_\nu(z) \) converges absolutely to an analytic function \( f(z) \) on a neighborhood of the origin; and

(b) For each fixed \( z \neq 0 \), the function of one complex variable, \( \zeta \to f(\zeta z) \), which is analytic for \( \zeta \) near the origin actually is an entire function, i.e. has an analytic continuation to all of \( \mathbb{C} \).

Prove that the formal power series actually converges absolutely and uniformly on compact subsets of \( \mathbb{C}^n \) to an entire function.

*Hint: The homogeneous polynomials satisfy an estimate of the form \( |p_\nu(z)| \leq C_\nu |z|^\nu \). What do the hypotheses and conclusion tell us about the size of the constants \( C_\nu \)?*