MATH 105 — FINAL EXAM

DEPARTMENT OF MATHEMATICS
University of Michigan

December 18, 2001

NAME: ___________________________    ID NUMBER: ___________________________

SIGNATURE: ________________________

INSTRUCTOR: ________________________    SECTION NO: ___________________________

1. This exam has 9 pages including this cover. There are 8 questions.
2. Use of books, notes, or scratch paper is not allowed. You may certainly use your calculator (but not its manual).
3. Show all of your work! Partial credit is available for many problems but can only be given if the graders understand your work. Be sure to explain your reasoning carefully. If you are basing your reasoning on a graph, then sketch the graph. Include units in your answers whenever appropriate.
4. One of the skills being tested in this exam is your ability to interpret detailed, precisely worded, directions. Be sure to read the directions carefully and do all that is asked.
5. Stay calm.

<table>
<thead>
<tr>
<th>PROBLEM</th>
<th>POINTS</th>
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</thead>
<tbody>
<tr>
<td>1</td>
<td>16</td>
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<tr>
<td>2</td>
<td>14</td>
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<td>3</td>
<td>8</td>
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<td>7</td>
<td>15</td>
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<td>8</td>
<td>8</td>
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<tr>
<td>TOTAL</td>
<td>100</td>
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</tbody>
</table>
1. Each of the figures below consists of a graph, and the equation describing that graph. Use the coordinates of the points given on each graph to find the values of the various constants. **For question 1 you need not show your reasoning.** (2 pts. each constant).

- **Linear Equation:**
  - $y=mx+c$
  - Points: $(1,7)$ and $(-3,-1)$
  - $m=\square$
  - $c=\square$

- **Quadratic Equation 1:**
  - $y=a(x-h)^2+k$
  - Points: $(0.5, -2.5)$
  - $a=\square$
  - $h=\square$
  - $k=\square$

- **Quadratic Equation 2:**
  - $y=k(x-a)(x-b)^2$
  - Points: $(3,0)$, $(6,0)$, and $(5,-8)$
  - $a=\square$
  - $b=\square$
  - $k=\square$
2. Each of the figures below consists of a graph, and the equation describing that graph. Use the coordinates of the points given on each graph to find the values of the various constants. **For question 2 you need not show your reasoning.** (2 pts. each constant).

![Graph 1](image1)

- **y = Asin(x) + D**
  - A = 
  - D = 

![Graph 2](image2)

- **y = ab^x**
  - a = 
  - b = 

![Graph 3](image3)

- **y = \( \frac{ax+b}{c-x} \)**
  - a = 
  - b = 
  - c = 

3. The table below describes the area $A = f(R)$ of a certain circular oil spill as a function of the radius, $R$ of that spill. The graph below describes the radius $R = g(t)$ of the oil spill as a function of the number of hours $t$, since the oil spill began.

<table>
<thead>
<tr>
<th>$R$</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>$f(R)$</td>
<td>0</td>
<td>3.14</td>
<td>12.57</td>
<td>28.27</td>
<td>50.27</td>
<td>78.54</td>
<td>113.1</td>
<td>153.94</td>
</tr>
</tbody>
</table>

(a) (2 pts.) Estimate $g(2)$.

(b) (2 pts.) Estimate $g^{-1}(2)$.

(c) (2 pts.) Estimate $f(g(2))$.

(d) (2 pts.) In one brief sentence describe the practical meaning of $f(g(2))$. 
4. A company believes that there is an exponential relationship between the consumer demand for its products and the price charged. Let $D(x)$ be the quantity per week demanded by customers at a unit price of $\$x$.

(a) (2 pts.) In the context of the question, give a brief interpretation of the meaning of the quantity $D(6)$. (You need not calculate the value of $D(6)$).

(b) (2 pts.) In the context of the question, give a brief interpretation of the meaning of the quantity $D^{-1}(60)$. (You need not calculate the value of $D^{-1}(60)$).

For parts (c) and (d), you will need the following additional information. When the price charged was $\$3$ per unit, the quantity demanded was 400 units per week. When the unit price was raised to $\$4$ per unit, the quantity demanded dropped to 300 units per week.

(c) (3 pts.) Find a formula for $D(x)$ in terms of $x$ assuming that $D(x)$ is exponential.

(d) (3 pts.) Currently the company can produce 350 units per week. What should the price per unit of the product be if the company wants to sell all 350 units?
Remember, if you are basing your reasoning on a graph, then sketch the graph.
5. A certain island has two harbors, harbor $P$ and harbor $S$. Let $D_P(t)$ denote the depth of the water in harbor $P$ and let $D_S(t)$ denote the depth of the water in harbor $S$. The time, $t$ is measured in hours after midnight on January the first, 2002.

(a) (6 pts.) Suppose that high tide occurs in harbor $S$ only at 3 a.m. and 3 p.m. on January the first, 2002, and that at high tide in harbor $S$ the depth of the water is 7 meters whereas at low tide it is 3 meters. If $D_S(t) = a \sin (bt) + c$, find $a$, $b$, and $c$.

$$a = \underline{\hspace{2cm}}$$

$$b = \underline{\hspace{2cm}}$$

$$c = \underline{\hspace{2cm}}$$

(b) (6 pts.) Due to overhead power cables at the entrance to harbor $P$, it can only be entered by boat when the water level is low — specifically, when the water’s depth is less than or equal to 5 meters. If $D_P(t) = 3 \sin \left(\frac{t}{h}\right) + 6$, what is the first time during January the first, 2002, that boats can enter harbor $P$?
6. During heavy rains, the water level in a local river must be closely monitored. The river will be at flood stage when it is 24 feet deep. At 12:00 noon a county agent measures the depth of the river and finds it to be 14 feet. Three hours later, she measures it again, and finds that the depth is 16 feet. To make predictions about flooding, she assumes that the river is rising steadily so that the depth is a linear function of time.

(a) (4 pts.) Write a formula for the depth of the river, \( d = f(t) \), a function of the time, \( t \), in hours since noon. (Let \( t = 0 \) be 12:00 noon).

(b) (3 pts.) What is the practical meaning of the slope of your formula?

(c) (4 pts.) Write a formula for the inverse function, \( f^{-1}(d) \) which gives time as a function of depth.

(d) (6 pts.) The agent is making a report on the 6:00 p.m. news. Fill in her predictions using the appropriate units.

"At this time, the river is [ ] deep, rising at a rate of [ ] and is expected to reach flood stage at [ ] tomorrow morning."
7. The following are short answer questions. **For question 7 you need not show your reasoning.**

(a) (2 pts.) Circle all of the types of functions which might have horizontal asymptotes?

<table>
<thead>
<tr>
<th>Sine</th>
<th>Cosine</th>
<th>Rational</th>
<th>Exponential</th>
<th>Logarithm</th>
</tr>
</thead>
</table>

(b) (4 pts.) By considering the following functions as \( x \) takes on larger and larger positive values, fill in the blanks.

\[
1.1^x \quad 100(1.09)^x \quad 10,000x^{10} \quad 1,000,000x \quad 1000\log(x)
\]

The fastest growing function is ________

The slowest growing function is ________

(c) (3 pts.) The point \((5, q)\) is on the graph of \(3x + 6 = y\). What is the value of \(q\)?

(d) (3 pts.) If you have a power function \(f(x) = kx^p\), what must be true of \(p\) for the graph to pass through the origin?

(e) (3 pts.) We say that the graph of a function changes direction if it goes from increasing to decreasing or from decreasing to increasing. Suppose that the graph of a polynomial changes direction three times. What can be said about the degree of this polynomial?
8. The amount of a quantity, $Q$, is a function of time, $t$. One description best fits each function. Decide which one, and write its letter in the corresponding blank. **For question 8 you need not show your reasoning.** (2 pts. each)

$$Q(t) = 400e^{-0.03t}$$

$$Q(t) = \frac{(3t+100)^2}{9t^2+100}$$

$$Q(t) = 100(0.03)^t$$

$$Q(t) = 100(t - 2)^2$$

(A) The quantity is originally 100 and eventually becomes close to 1.

(B) The quantity is originally 100, but it decreases to zero in two years.

(C) The quantity is originally 400 and it decreases at the continuous rate of 3% per year.

(D) The quantity is originally 100 and it decreases at the constant rate of 3% per year.

(E) The quantity is originally 400, but it decreases to zero in two years.

(F) The quantity is originally 100 and eventually becomes close to 30.

(G) The quantity starts at 100 and it decreases at the constant rate of 97% per year.

(H) The quantity is originally 400 and it decreases at the constant rate of 3% per year.