1. (17 points) Consider the region bounded by the curve \( y = \sqrt{x} \), the \( x \)-axis, and the line \( x = 2 \). A metal slug has the shape of the solid obtained by rotating this region about the \( x \)-axis. The density of the slug is given by \( \rho(x) = 1 + 3x \) grams/cm\(^3\) for \( 0 \leq x \leq 2 \). Find the exact total mass of the slug. The units on both the \( x \) and \( y \) axes are centimeters. Clarity please.

**Solution.** The part of this solid between \( x \) and \( x + \Delta x \) with \( \Delta x \) small is a disk of radius \( \approx \sqrt{x} \), thickness \( \Delta x \) and nearly constant density \( 1 + 3x \). Therefore

\[
\Delta \text{Volume} \approx \pi (\sqrt{x})^2 \Delta x \approx \pi x \Delta x, \quad \Delta \text{Mass} \approx \text{density} \times \Delta \text{Volume} \approx (1 + 3x)\pi x \Delta x.
\]

Dividing the interval \( 0 \leq x \leq 2 \) into \( N \) equal subintervals and summing the masses yields

\[
\text{Mass} \approx \sum_i \Delta \text{Mass}_i \approx \sum_i (1 + 3x_i)\pi x_i \Delta x.
\]

Passing to the limit yields the exact expression

\[
\text{Mass} = \int_0^2 (1 + 3x)\pi x \, dx = \pi \int_0^2 x + 3x^2 \, dx.
\]

An antiderivative is \( x^2/2 + x^3 \) so the fundamental theorem of calculus implies that

\[
\text{Mass} = \pi (x^2/2 + x^3)|_0^2 = \pi (2^2/2 + 2^3) = 10\pi \text{ grams}.
\]

2. (5+5+10 points) In a booming economy money is deposited at a rate of \( e^{0.03t} \) trillion dollars per year for \( 0 \leq t \leq 10 \) with \( t \) measured in years. Assume that money earns interest at a continuously compounded rate of 5% per year. Recall that this means that after \( y \) years one dollar grows to \( e^{0.05y} \) dollars.

**a.** Without using an integral, estimate approximately how many dollars are deposited in the time interval \( 2 \leq t \leq 2.001 \)?

**Solution.** The rate of deposit during this brief interval is approximately \( e^{0.03 \times 2} \) so the deposit is approximately equal to

\[
e^{0.06} \Delta t = e^{0.06}(0.001) \text{ trillion dollars}.
\]

**b.** Find the approximate present value at \( t = 0 \) of the deposit computed in part **a.** Again this answer does not involve an integral.

**Solution.** The present value at \( t = 0 \) is a value two years earlier, so is smaller by a factor \( e^{-0.05 \times 2} \) so

\[
\left( \text{present value at } t = 0 \right) \approx e^{-0.1} e^{0.06}(0.001) = e^{-0.04}(0.001) \text{ trillion dollars}.
\]

**c.** Find the exact present value at \( t = 0 \) of the ten year deposit stream.

**Solution.** In the short interval from \( t \) to \( t + \Delta t \) approximately

\[
e^{0.03t} \Delta t
\]

trillion dollars are deposited.
Show work and explain your reasoning clearly.

The present value at time $t = 0$ of this deposit is a value $t$ years earlier so is approximately equal to

$$e^{-0.05t} e^{0.03t} \Delta t.$$ 

Summing these small contributions and passing to the limit of finer and finer subdivisions yields the exact value

$$\int_0^{10} e^{-0.05t} e^{0.03t} dt = \int_0^{10} e^{-0.02t} dt.$$ 

An antiderivative is $e^{-0.02t}/(-0.02)$, so the fundamental theorem of calculus yields the exact value

$$e^{-0.02t}/(-0.02)|^0_{10} = 50(1 - e^{-0.2}) \text{ trillion dollars.}$$

3. (5+5+5+5 points) Wheat is treated with fertilizer and the yield per plant is studied. The cumulative distribution function for the yields per plant in hundredths of a bushel is plotted in the graph below.

\[\text{Graph Image}\]

a. Find the lowest and highest observed yields? Explain very briefly how you know this.

**Solution.** The distribution function has value 0 at 2 hundredths of a bushel per plant and value 1 at 13 hundredths of a bushel per plant. Therefore the probability of a measurement lower than 2 or higher than 13 is equal to zero. (Remark. This is all that is required for full credit.)

At the same time for any interval $2 \leq x \leq 2 + \Delta x$ the probability of yields in this range is equal to $P(2 + \Delta x) > 0$. Thus 2 is the lowest observed value. Similarly the probability of a measurement greater than 13 $- \Delta x$ is equal to $1 - P(13 - \Delta x) > 0$, so 13 is the largest observed value.

b. Find the median yield. Explain very briefly. Recall that the median is that yield so that exactly half of the sample has yield lower than this value.

**Solution.** The cumulative distribution function $P$ in the graph attains the value 0.5 exactly when the independent variable has value equal to 7 hundredths of a bushel. This means that the probability of a yield lower than 7 hundredths is exactly equal to 0.5 so the median is exactly 7 hundredths of a bushel.

c. At what yield(s) does the probability density function have its largest value? Explain very briefly how you know this.

**Solution.** The probability density function is the derivative of the cumulative distribution function, so we are looking for the point(s) on the graph where the slope is greatest. This occurs near 4.5 hundredths of a bushel.

d. If you pick a plant at random, then is the probability that the yield lies between 6 and 7 larger or smaller than 1/6? Explain very briefly.

**Solution.** The probability in question is equal to $P(7) - P(6)$ which is the gain in height in the cumulative distribution when the independent variable increases from 6 to 7. From the graph it is clear that $P(6) > 2/6$ and $P(7) = 3/6$ so the gain in height is less than 1/6.
4. (7+5 points) a. Find the first three nonzero terms in the Taylor series of
\[ f(x) = \frac{1}{\sqrt{1 + x^2}}, \]
centered at \( x = 0. \)

**Solution.** Write the function as \((1 + u)^{-1/2}\) with \( u = x^2. \) Use the binomial expansion
\[
(1 + u)^{-1/2} \sim 1 + pu + \frac{p(p - 1)}{1 \times 2}u^2 + \cdots \sim 1 + (-1/2)u + \frac{(-1/2)(-3/2)}{1(2)}u^2 + \cdots .
\]
Plug in \( u = x^2 \) to find
\[
(1 + x^2)^{-1} \sim 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 + \cdots ,
\]
and the Taylor polynomial with three nonvanishing terms
\[
f \approx P_4 = 1 - \frac{1}{2}x^2 + \frac{3}{8}x^4 \]
is determined.

b. Use Taylor polynomials to determine which of the two functions
\[ f(x) = \frac{1}{\sqrt{1 + x^2}}, \quad \text{and} \quad g(x) = \cos x, \]
is larger for small positive values of \( x. \)

**Solution.** The fourth order Taylor polynomial of \( g(x) \) is found using the Taylor series on the cover page for \( \cos x \) and is equal to
\[
g \approx 1 - x^2/2 + x^4/4! = 1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 .
\]
Subtracting the fourth order Taylor polynomials shows that near \( x = 0, \)
\[
f(x) - g(x) \sim \left( \frac{3}{8} - \frac{1}{24} \right)x^4 .
\]
As this is positive, we conclude that \( f - g \) is positive near \( x = 0 \) and therefore that \( f \) is larger than \( g \) near \( x = 0. \)

5. (5+4+4+4 points) a. A function \( f(x) \) is periodic with period \( 2\pi \) and its values for \(-\pi \leq x \leq \pi \) are given by
\[
f(x) = \begin{cases} 
0 & \text{for} \quad -\pi \leq x < 0 , \\
1 & \text{for} \quad 0 \leq x < \pi/3 , \\
0 & \text{for} \quad \pi/3 \leq x \leq \pi .
\end{cases}
\]
Sketch the values of \( f \) for \(-3\pi \leq x \leq 3\pi \) on the axes below. Be sure to label the coordinates of
Show work and explain your reasoning clearly.

important points.

b. The Fourier series of $f(x)$ is

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos nx + b_n \sin nx).$$

Find the Fourier approximation which consists of the constant term and the first harmonic, that is the constant term and the terms with $n = 1$. You must compute the exact values of $a_0$, $a_1$, and $b_1$. Useful formulas are given on the cover page of this exam.

Solution.

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) \, dx = \frac{1}{2\pi} \int_{0}^{\pi/3} 1 \, dx = \frac{1}{2\pi} \frac{\pi}{3} = \frac{1}{6}.$$

$$a_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos x \, dx = \frac{1}{\pi} \int_{0}^{\pi/3} 1 \cos x \, dx = \frac{1}{\pi} (\sin x)|^{\pi/3}_0$$

$$= \frac{1}{\pi} \left( \sin \frac{\pi}{3} - \sin 0 \right) = \frac{1}{\pi} \left( \frac{\sqrt{3}}{2} - 0 \right) = \frac{\sqrt{3}}{2\pi}.$$

$$b_1 = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin x \, dx = \frac{1}{\pi} \int_{0}^{\pi/3} 1 \sin x \, dx = \frac{1}{\pi} (-\cos x)|^{\pi/3}_0$$

$$= \frac{1}{\pi} \left( - \cos \frac{\pi}{3} + \cos 0 \right) = \frac{1}{\pi} (-1/2 + 1) = \frac{1}{2\pi}.$$

The approximation including the constant term and the leading harmonic is

$$f(x) \approx a_0 + a_1 \cos x + b_1 \sin x = \frac{1}{6} + \frac{\sqrt{3}}{2\pi} \cos x + \frac{1}{2\pi} \sin x.$$

6. (8+6 points) a. Find parametric equations for the quarter circle of radius equal to 2 with center at $x = 0$, $y = 0$. The parametrization should start at the point $(0, 2)$ and travel clockwise to $(2, 0)$. 

$$y$$

$$(0,2)$$

$$(2,0)$$
**Solution.** The circle of radius two traversed clockwise is parametrized by

\[ x = 2 \cos t, \quad y = -2 \sin t. \]

This parametrization passes through (2, 0) at \( t = -\pi/2 \) and (0, 2) at \( t = 0 \).

The desired parametrization results from a shift of the time by \( \pi/2 \) units,

\[ x = 2 \cos(t - \pi/2), \quad y = -2 \sin(t - \pi/2), \quad 0 \leq t \leq \pi/2. \]

**b.** Find parametric equations for the quarter circle of radius equal to 2 with center at \( x = 1, y = 3 \). The parametrization should start at the point (1, 5) and travel clockwise to (3, 3).

**Solution.** It suffices to simply shift the \( x \) and \( y \) coordinates of the preceding curve by 1 and 3 respectively to find

\[ x = 2 \cos(t - \pi/2) + 1, \quad y = -2 \sin(t - \pi/2) + 3. \]