1. Useful formulas are provided below. There are 7 questions.
2. Include units in your answers whenever appropriate.
3. You are allowed three sides of 3 by 5 cards of notes. You may also use your calculator.
4. If you use an approximate integration program or an approximate ordinary differential equation solver on your calculator, you must indicate which method(s) you use and also the number of subintervals.
5. Show your work and explain your reasoning clearly.

Some Formulas

Continuous growth at a rate of $r\%$ annually and $r\%$ interest compounded continuously are both expressions for exponential growth $C e^{rt/100}$ with constant $C$.

Geometric Series

$$a + ar + ar^2 + ar^3 + \ldots ar^M = \frac{a(1 - r^{M+1})}{1 - r}, \quad \{r \neq 1\}.$$  

General Taylor series

$$f(x) \sim f(a) + f'(a)(x - a) + \frac{f''(a)}{2!}(x - a)^2 + \frac{f'''(a)}{3!}(x - a)^3 + \ldots.$$  

Specific Taylor Series centered at $a = 0$

$$\ln(1 + x) \sim x - \frac{x^2}{2} + \frac{x^3}{3} \ldots, \quad (1 + x)^p \sim 1 + px + \frac{p(p - 1)}{1 \cdot 2}x^2 + \frac{p(p - 1)(p - 2)}{1 \cdot 2 \cdot 3}x^3 \ldots,$$

$$e^x \sim 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} \ldots, \quad \sin x \sim x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots, \quad \cos x \sim 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots.$$  

Taylor Series Error Formula

$$|E_n(x)| \leq \frac{M |x - a|^{n+1}}{(n + 1)!}, \quad M = \max |f^{(n+1)}| \text{ between } a \text{ and } x.$$
Fourier Series of functions with period $p$

$$f = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi nx}{p} + b_n \sin \frac{2\pi nx}{p}),$$

$$a_0 = \frac{1}{p} \int_{-p/2}^{p/2} f(x) \, dx = \frac{1}{p} \int_{0}^{p} f(x) \, dx,$$

and for $n \geq 1$,

$$a_n = \frac{2}{p} \int_{-p/2}^{p/2} f(x) \cos \frac{2\pi nx}{p} \, dx = \frac{2}{p} \int_{0}^{p} f(x) \cos \frac{2\pi nx}{p} \, dx,$$

$$b_n = \frac{2}{p} \int_{-p/2}^{p/2} f(x) \sin \frac{2\pi nx}{p} \, dx = \frac{2}{p} \int_{0}^{p} f(x) \sin \frac{2\pi nx}{p} \, dx.$$

Energy

$$\text{Energy} = \frac{2}{p} \int_{-p/2}^{p/2} |f(x)|^2 \, dx = \frac{2}{p} \int_{0}^{p} |f(x)|^2 \, dx = 2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2).$$

1. (5+8+7 points) The concentration of pollutant in Gledhill Lake is changing at the rate $r(t)$ shown in the graph below.

At $t = 1988$, the left hand limit in the graph above, the concentration was 140 milligrams/Liter (mg/L).

**a.** On what interval(s) was the concentration of pollutant increasing?

**b.** When was the lake most polluted and when was it least polluted? What was the concentration of pollutant at those times?

**c.** Sketch a graph of the pollutant’s concentration from 1988 to 2000.
2. (12 points) The results from LEFT(12), RIGHT(12), TRAP(12) and MID(12) are given in the table for four functions \( f(x) \), \( g(x) \), \( h(x) \) and \( k(x) \).

<table>
<thead>
<tr>
<th></th>
<th>( f(x) )</th>
<th>( g(x) )</th>
<th>( h(x) )</th>
<th>( k(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>LEFT(12)</td>
<td>1.738</td>
<td>1.537</td>
<td>1.285</td>
<td>1.484</td>
</tr>
<tr>
<td>RIGHT(12)</td>
<td>1.905</td>
<td>1.515</td>
<td>1.327</td>
<td>1.401</td>
</tr>
<tr>
<td>TRAP(12)</td>
<td>1.822</td>
<td>1.526</td>
<td>1.306</td>
<td>1.443</td>
</tr>
<tr>
<td>MID(12)</td>
<td>1.820</td>
<td>1.526</td>
<td>1.307</td>
<td>1.442</td>
</tr>
</tbody>
</table>

Under each of the graphs indicate the name of the corresponding function \( f(x) \), \( g(x) \), \( h(x) \) or \( k(x) \). No explanation is needed.

3. (8+12 points) In the late 1700’s, 4000 convicts arrived annually at Port Jackson, Australia at the beginning of the year. As residents of Port Jackson, they endured many hardships so that in each year only 60% of the convict population survived. The birth rate among the residents was negligible.

\textbf{a.} The first fleet of convicts arrived in 1788. Find the number of convicts resident in Port Jackson at the beginning of 1791, immediately after that year’s arrivals land.

\textbf{b.} The governor claimed that at this rate, the convict population would never exceed 12 000. Was he correct? Justify your answer.

4. (8 + 12 points) To cure a certain disease, the concentration of the drug Anadine in the patient’s body should be maintained at the constant concentration 0.05 mg/Liter for at least 3 days. When patients are treated with this drug, their body destroys it at rate which is proportional to the concentration \( c(t) \). The constant of proportionality is equal to 0.3 and the time \( t \) is measured in hours.

The treatment has two stages.

\textbf{Stage 1.} Anadine is administered at a rate of 0.6 mg/hour till the concentration reaches 0.05 mg/L.

\textbf{Stage 2.} The rate \( r \) at which Anadine is administered is changed so that the concentration in the body remains constant at 0.05 mg/L.

Assume that the volume of the part of a human body where the drug will circulate is 35 Liters and that the drug mixes instantaneously.

\textbf{a.} Find a differential equation satisfied by the concentration \( c(t) \) during stage 1.

\textbf{b.} Using this equation, determine how long stage 1 lasts.
Show work and explain your reasoning clearly.

5. (20 points) The next problems concern the ordinary differential equation
\[ \frac{dx}{dt} = f(x) , \]
where the function \( f(x) \) is given in the graph below.

![Graph of f(x)](image)

a. Use two steps of Euler’s method with \( \Delta t = .1 \) to determine, by hand, an approximate value of \( x(0.2) \) for the solution \( x(t) \) of the differential which satisfies \( x(0) = 0.3 \) initially.

b. Determine all the equilibria of the differential equation.

c. Sketch the slope field of the differential equation on the axes provided.

![Axes with slope field](image)

d. For the solution satisfying \( x(0) = .3 \), sketch the solution curve on the same axes. What is the behavior of this solution as \( t \to \infty \)?

6. (6+6+2+6 points) A glider descends to the landing strip in Bieske Field. The glider’s path is given by
\[ x(t) = 5000t , \quad y(t) = 500(t - 2)^2 . \]
Here \( x(t) \) and \( y(t) \) are measured in feet and the time \( t \) is measured in seconds. \( y(t) \) is the altitude above the field and \( x(t) \) is the horizontal distance from the edge of the Field.

a. Find the glider’s speed at time \( t = 0 \).

b. Find the line which is tangent to the glider’s path at time \( t = 0 \).

c. Find the time when the glider reaches the ground.

d. Write an expression for the exact distance the glider travels between \( t = 0 \) and the time it lands. Do not evaluate.
7. (7+8+5 points) Yellow birch and jack pine trees compete for sunlight and soil nutrients in Nameez Forest. Let $B$ denote the number of yellow birch trees in the forest, and let $P$ denote the number of jack pines. Assume that the tree populations in the forest are modeled by the system of differential equations:

\[
\frac{dB}{dt} = B(400 - B - P) \\
\frac{dP}{dt} = P(600 - 2B - P)
\]

a. Find the nullclines and equilibria of the system.

b. Sketch the phase plane, showing the nullclines, the equilibria, and the direction of the trajectories in each region.

c. If there are 100 yellow birch trees and 300 jack pines in the forest today, what does the model predict about the tree populations in the long run?