1. Do not open this exam until you are told to begin.

2. This exam has 10 pages including this cover. There are 8 questions.

3. Do not separate the pages of the exam. If any pages do become separated, write your name on them and point them out to your instructor when you turn in the exam.

4. Please read the instructions for each individual exercise carefully. One of the skills being tested on this exam is your ability to interpret questions, so instructors will not answer questions about exam problems during the exam.

5. Show an appropriate amount of work for each exercise so that the graders can see not only the answer but also how you obtained it. Include units in your answers where appropriate.

6. Calculators are not to be used for integration or antidifferentiation. If an integral problem lacks sufficient supporting work, zero credit will be given even if the answer is correct.

7. You may use your calculator, except as noted above. You are also allowed 2 sides of a 3 by 5 note card.

8. If you use graphs or tables to obtain an answer, be certain to provide an explanation and sketch of the graph to make clear how you arrived at your solution.

9. Please turn off all cell phones and other sound devices, and remove all headphones.

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1. (18 points) Short answer questions. No Partial Credit. Show your work. Do not use your calculator for finding anti-derivatives or for computing definite integrals.

(a) (3 points) Find the exact value of the average value of the function \( f(x) = e^{2x} \) on the interval \( 0 \leq x \leq 3. \)

\[
\overline{f} = \frac{1}{3 - 0} \int_0^3 e^{2x} \, dx \\
= \frac{1}{3} \left[ \frac{e^x}{2} \right]_0^3 \\
= \frac{e^6 - 1}{6}
\]

(b) (3 points) Does the integral \( \int_2^\infty \frac{dx}{x^3} \) converge or diverge? If it converges, compute its value.

\[
\int_2^\infty \frac{dx}{x^3} = \lim_{T \to \infty} \int_0^T \frac{dx}{x^3} \\
= \lim_{T \to \infty} \left[ -\frac{1}{2x^2} \right]_2^T \\
= \lim_{T \to \infty} \left( \frac{1}{8} - \frac{1}{2T^2} \right) = \frac{1}{8}
\]

(c) (3 points) If \( \int_0^2 f(t) \, dt = 8, \) find \( \int_0^1 f(2t) \, dt. \)

Set

\[
\begin{align*}
    u &= 2t \\
    du &= 2dt
\end{align*}
\]

\[
    u(0) = 0 \\
    u(1) = 2
\]

Now we have:

\[
\int_0^1 f(2t) \, dt = \int_{u(0)}^{u(1)} f(u) \frac{du}{2} \\
= \frac{1}{2} \int_0^2 f(u) \, du \\
= \frac{1}{2} \int_0^2 f(t) \, dt = 4
\]

(continued on next page)
(d) **(3 points)** Determine the derivative \( \frac{d}{dx} \int_x^{x^2} \frac{1}{\sin t} \, dt \).

\[
\frac{d}{dx} \int_x^{x^2} \frac{1}{\sin t} \, dt = \frac{d}{dx} \left( \int_0^{x^2} \frac{dt}{\sin t} - \int_0^x \frac{dt}{\sin t} \right) \\
= \frac{d}{dx} \left( \int_0^x \frac{du}{\sin u} \right) \frac{du}{dx} - \frac{1}{\sin x} \\
= \frac{2x}{\sin(x^2)} - \frac{1}{\sin x}
\]

where we set \( u(x) = x^2 \) and used the chain rule for differentiation.

(e) **(3 points)** If \( \frac{2}{x^2 - 1} = \frac{A}{x + 1} + \frac{B}{x - 1} \), find \( A \) and \( B \).

The right hand side simplifies to:

\[
\frac{A}{x + 1} + \frac{B}{x - 1} = \frac{A(x - 1) + B(x + 1)}{(x + 1)(x - 1)} = \frac{(A + B)x + (B - A)}{x^2 - 1}
\]

Comparing the the left hand side, we see that the following must hold:

\[
A + B = 0 \quad \text{and} \quad B - A = 2
\]

Therefore

\[
A = -1 \\
B = 1
\]

(f) **(3 points)** If \( f(x) \) is concave down on the interval \( 0 \leq x \leq 6 \), what type of approximation method will give you an overestimate of the integral \( \int_0^6 f(x) \, dx \)? Circle ONE answer.

Midpoint Rule
2. **(8 points)** A certain drug is being continuously injected into a patient’s bloodstream at the rate $J(t)$ mL/hr. The drug is being absorbed by the body from the bloodstream at the rate $A(t)$ mL/hr. At time $t = 0$, there is no drug in the patient’s bloodstream.

(a) **(2 points)** Determine the time at which the *net rate of increase* of the drug in the patient’s bloodstream was the greatest.

$t = 2$

(b) **(2 points)** At what time was the greatest amount of the drug present in the patient’s bloodstream?

$t = 5$

(c) **(2 points)** The area enclosed by the two curves between $t = 0$ and $t = 5$ is 130. What is the practical interpretation of this area? Include units.

There has been a net increase of 130 mL of the drug to the patient’s bloodstream over the first 5 hours. Since none of the drug was initially present in the bloodstream, this implies that the patient had 130 mL of the drug in his/her bloodstream at $t = 5$ hours.

(d) **(2 points)** If $\int_0^5 J(t) \, dt = 242$, how much of the drug was absorbed from the bloodstream into the patient’s body during the first 5 hours of treatment? Include appropriate units with your answer.

The known area gives us the fact that:

$$\int_0^5 J(t) \, dt - \int_0^5 A(t) \, dt = 130 \text{ mL}$$

Solving for the amount absorbed from the bloodstream into the patient’s body, we find:

$$\int_0^5 A(t) \, dt = \int_0^5 J(t) \, dt - 130 = 112 \text{ mL}$$
3. (12 points) Consider the graph of \( f(x) \), \( g(x) \), \( h(x) \), \( 1/x \), and \( 1/x^2 \) below.

You may assume that

\[
\begin{align*}
    h(x) &< \frac{1}{x^2} < g(x) < \frac{1}{x} < f(x) \quad \text{for all } x > 1 \\
    f(x) &< \frac{1}{x} < g(x) < \frac{1}{x^2} < h(x) \quad \text{for all } 0 < x < 1
\end{align*}
\]

Determine the convergence or divergence of each integral where possible. Circle ONE for each.

(a) (3 points) \( \int_1^\infty f(x) \, dx \).
\( \int_1^\infty \frac{dx}{x} \) diverges and \( f(x) > \frac{1}{x} \) for all \( 1 < x < \infty \), therefore the integral diverges.

(b) (3 points) \( \int_1^\infty g(x) \, dx \).
We have \( \frac{1}{x^2} < g(x) < \frac{1}{x} \) for all \( 1 < x < \infty \). So the integral of \( g(x) \) is smaller than some divergent integral but larger than some convergent integral. Thus there is Not Enough Information to determine its convergence.

(c) (3 points) \( \int_1^\infty h(x) \, dx \).
\( h(x) < \frac{1}{x^2} \) for \( 1 < x < \infty \). Hence the integral of \( h(x) \) converges.

(d) (3 points) \( \int_0^1 f(x) \, dx \).
\( f(x) < \frac{1}{x} \) on the interval \( 0 < x < 1 \). However, \( \int_0^1 \frac{dx}{x} \) diverges, thus there is Not Enough Information.
4. (12 points) Consider the graph of the function $f(s)$ pictured below. The domain of $f(s)$ is $0 \leq s \leq 6$.

Let $F(x)$ be the function defined by $F(x) = \int_0^x f(s) \, ds$.

(a) (4 points) List all intervals over which $F(x)$ is concave up.

$F(x)$ is concave up when its derivative $F'(x) = f(x)$ is increasing.

$$0 < x < 1 \quad \text{and} \quad 4 < x < 6$$

(b) (4 points) List all intervals over which $F(x)$ is increasing.

$F(x)$ is increasing when $F'(x) = f(x)$ is positive.

$$0 < x < 3 \quad \text{and} \quad 5 < x < 6$$

(c) (4 points) On the axes provided below, sketch an accurate graph of $F(x)$. Include an appropriate scale on the $y$-axis.

The red dots mark inflection points. The $y$-axis should be fairly close to what is depicted below. $F(0) = 0$. The graph should be increasing/decreasing and concave up/concave down on the appropriate intervals. There should be a local maximum at $x = 3$ and a local minimum at $x = 5$. 
5. **(16 points)** Let $LEFT(N)$ and $RIGHT(N)$ be the left and right hand Riemann Sums of the integral 

$$
\int_{0}^{\pi/2} \sin(x) \, dx
$$

with $N$ subintervals.

(a) **(4 points)** Find the numerical value of $RIGHT(3)$. Give at least 3 decimal places and show your work.

$$
RIGHT(3) = \frac{\pi}{6} \left( \sin \left( \frac{\pi}{6} \right) + \sin \left( \frac{\pi}{3} \right) + \sin \left( \frac{\pi}{2} \right) \right) \\
= \frac{\pi}{6} \left( \frac{1}{2} + \frac{\sqrt{3}}{2} + 1 \right) \\
\approx 1.2388
$$

(b) **(4 points)** On the diagram provided, sketch the area that you would compute when calculating $LEFT(3)$.

(c) **(3 points)** Which Riemann Sum ($LEFT(3)$ or $RIGHT(3)$) will give an overestimate of the definite integral? Illustrate your answer with an appropriate sketch.

$RIGHT(3)$ will give an overestimate.

(d) **(5 points)** You are told that for a certain number $N$ of subintervals, $LEFT(N) = 0.98823$ and $\Delta x = 0.02344$. Find $RIGHT(N)$ accurate to five decimal places.

In general, we have that:

$$
RIGHT(N) - LEFT(N) = \Delta x (f(b) - f(a))
$$

We have that

$$
f(b) = \sin \left( \frac{\pi}{2} \right) = 1 \\
f(a) = \sin(0) = 0
$$

Hence, we can solve for

$$
RIGHT(N) = 1.01167
$$
6. **(12 points)** Determine whether each of the following integrals converges or diverges. If an integral diverges, explain how you know this. If an integral converges, compute its value. Do not use your calculator. Use appropriate limit notation when evaluating convergent improper integrals. Show all work.

(a) **(6 points)** \( \int_0^\infty te^{-\alpha t^2} \, dt \) where \( \alpha > 0 \) is a positive constant.

We first compute \( \int_0^M te^{-\alpha t^2} \, dt \) using \( u \)-substitution:

\[
\begin{align*}
\quad u(t) &= \alpha t^2 \\
\quad du &= 2\alpha t \, dt \\
\quad u(0) &= 0 \\
\quad u(M) &= \alpha M^2 \\
\end{align*}
\]

We now have:

\[
\int_0^M te^{-\alpha t^2} \, dt = \frac{1}{2\alpha} \int_0^{\alpha M^2} e^{-u} \, du = \frac{-1}{2\alpha} e^{-u} \bigg|_0^{\alpha M^2} = \frac{1}{2\alpha} - \frac{e^{-\alpha M^2}}{2\alpha}
\]

We thus have

\[
\int_0^\infty te^{-\alpha t^2} \, dt = \lim_{M \to \infty} \int_0^M te^{-\alpha t^2} \, dt = \lim_{M \to \infty} \left[ \frac{1}{2\alpha} - \frac{e^{-\alpha M^2}}{2\alpha} \right] = \frac{1}{2\alpha}
\]

(b) **(6 points)** \( \int_0^\infty te^{-\beta t} \, dt \) where \( \beta > 0 \) is a positive constant.

We first compute \( \int_0^M te^{-\beta t} \, dt \) using Integration by Parts.

\[
\begin{align*}
\quad u &= t \\
\quad du &= dt \\
\quad dv &= e^{-\beta t} \, dt \\
\quad v &= \frac{e^{-\beta t}}{-\beta}
\end{align*}
\]

Hence

\[
\int_0^M te^{-\beta t} \, dt = \left. \frac{te^{-\beta t}}{-\beta} \right|_0^M + \frac{1}{\beta} \int_0^M e^{-\beta t} \, dt = \frac{-Me^{-\beta M}}{\beta} - \frac{e^{-\beta M}}{\beta^2} + \frac{1}{\beta^2}
\]

We thus have:

\[
\int_0^\infty te^{-\beta t} \, dt = \lim_{M \to \infty} \int_0^M te^{-\beta t} \, dt = \lim_{M \to \infty} \left( \frac{-Me^{-\beta M}}{\beta} - \frac{e^{-\beta M}}{\beta^2} + \frac{1}{\beta^2} \right) = \frac{1}{\beta^2}
\]
7. **(12 points)** Below you will find a graph of the function \( g(x) \), and a table containing some information of \( f(x) \). **You may also assume that** \( f'(x) \) **is even.**

<table>
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<tr>
<th>( x )</th>
<th>-3</th>
<th>-2</th>
<th>-1</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>8</td>
<td>9</td>
<td>11</td>
<td>16</td>
<td>21</td>
<td>23</td>
<td>24</td>
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</tbody>
</table>

(a) **(3 points)** Assuming \( f'(x) \) is even, explain why \( \int_{-3}^{3} g(x)f'(x) \, dx = 0 \).

The product of an even and an odd function is odd, thus \( g(x) \cdot f'(x) \) is odd. The integral of an odd function over the interval \((−a, a)\) is zero.

(b) **(4 points)** Using Integration by Parts, find the exact value of \( \int_{-3}^{3} f(x)g'(x) \, dx \).

\[
\int_{-3}^{3} f(x)g'(x) \, dx = f(x)g(x)|_{-3}^{3} - \int_{-3}^{3} g(x)f'(x) \, dx \\
= f(3)g(3) - f(-3)g(-3) = 32
\]

(c) **(5 points)** Using \( u \)-substitution, find the exact value for \( \int_{-3}^{3} f'(g(x))g'(x) \, dx \).

Set

\[
\begin{align*}
  u(x) &= g(x) \\
  du &= g'(x)dx \\
  u(-3) &= g(-3) = -1 \\
  u(3) &= g(3) = 1
\end{align*}
\]

Hence

\[
\int_{-3}^{3} f'(g(x))g'(x) \, dx = \int_{-1}^{1} f'(u) \, du = f(1) - f(-1) = 10
\]
8. **(10 points)** Suppose you are the supervisor of a diamond mine. You wish to determine how long each shift should last in order to maximize the productivity of the mine. Let $D(t)$ be the rate at which the miners discover diamonds, measured in Diamonds per hour, $t$ hours after the start of each shift. $D(t)$ is plotted below.

![Graph of D(t) with t-hours on the x-axis and Diamonds/hr on the y-axis.]

(a) **(3 points)** Write an expression for the average rate of diamond discovery, $a(M)$, for the first $M$ hours of the work shift.

$$a(M) = \frac{1}{M} \int_0^M D(t) \, dt$$

(b) **(4 points)** At the time $M$, when $a(M)$ reaches its maximum, $a(M) = D(M)$. Derive this relation using mathematics. *Hint: First compute $a'(M)$.*

$$a'(M) = \frac{1}{M} \frac{d}{dM} \left( \int_0^M D(t) \, dt \right) - \frac{1}{M^2} \int_0^M D(t) \, dt$$

$$= \frac{1}{M} D(M) - \frac{1}{M^2} \int_0^M D(t) \, dt$$

When $a(M)$ reaches a maximum, we can assume $a'(M) = 0$. (We are not interested in solutions at the endpoint $M = 0$). $a'(M) = 0$ implies that:

$$\frac{1}{M} D(M) = \frac{1}{M^2} \int_0^M D(t) \, dt$$

$$\frac{1}{M} D(M) = \frac{1}{M} a(M)$$

Thus $D(M) = a(M)$.

(c) **(3 points)** Approximate how long each shift should last, accurate to the nearest half hour, in order to maximize the average rate of diamond discovery.

Each shift should last for approximately 5.5 hours.

We see in the graph that the average value $a(5.5)$ should be equal to the height of the indicated rectangle (BECAUSE the area of the rectangle is approximately equal to $\int_0^{5.5} D(t) \, dt$). Thus the average value of $D(t)$ over the interval $(0, 5.5)$ is approximately equal to $D(5.5)$. This is the condition for a critical point in $A(M)$.