Problem 1. \((5+5+4+4+2 = 20 \text{ points})\) Let \(A = (2, -5, 6), B = (3, -5, 7),\) and \(C = (3, -6, 8).\)

(a) Compute \(\vec{A}B \times \vec{A}C\) and \(\vec{A}B \cdot \vec{A}C.\)

\[
\vec{A}B = \langle 1, 0, 1 \rangle \\
\vec{A}C = \langle 1, -1, 2 \rangle \\
\vec{A}B \times \vec{A}C = \langle 1, -1, -1 \rangle \\
\vec{A}B \cdot \vec{A}C = 3
\]

(b) Find the angle between the vectors \(\vec{A}C\) and \(\vec{A}B.\)

Call the angle \(\theta.
\]

\[
\theta = \arccos \left( \frac{\vec{A}B \cdot \vec{A}C}{|\vec{A}B||\vec{A}C|} \right) = \arccos \left( \frac{3}{\sqrt{2}\sqrt{16}} \right) \\
= \arccos \left( \frac{\sqrt{3}}{2} \right) = \frac{\pi}{6}
\]

Answer: \(\frac{\pi}{6}\)
(c) Find a parametric equation for the line which contains the point \( B \) and is parallel to the vector \( \overrightarrow{AC} \)

\[
\vec{r}(s) = \langle 3+s, -5-s, 7+2s \rangle
\]

Answer: (c)

(d) Find the equation for the plane containing the points \( A, B, \) and \( C \).

The plane of interest contains \((2, -5, 6)\) and is orthogonal to \( \overrightarrow{AB} \times \overrightarrow{AC} \). Use standard form.

Answer: \( x - y - z - 1 = 0 \)

(e) True or False: The line defined in part (c) must lie in the plane defined in part (d). Justify your answer.

We check:

\[
(3+s) - (-5-s) - (7+2s) - 1 \overset{?}{=} 0
\]

Yes.

Answer: True.
Problem 2.  \(5 + 5 = 10\) points) Consider the curve \(C\) parametrized by
\[
\mathbf{r}(\theta) = (\sin^3(\theta), \cos^3(\theta), \sin^2(\theta))
\]
for \(0 \leq \theta \leq \pi/2\).

(a) Provide a parametric equation for the line tangent to \(C\) at the point \((\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{1}{2})\).

Note: \(\mathbf{r}'(\pi/4) = \left< \frac{1}{2\sqrt{2}}, \frac{1}{2\sqrt{2}}, \frac{1}{2} \right>\). We need to calculate \(\mathbf{r}'(\pi/4)\).

Since \(\mathbf{r}'(\theta) = \langle 3 \sin^2(\theta) \cos(\theta), -3 \cos^2(\theta) \sin(\theta), 2 \sin(\theta) \cos(\theta) \rangle\)
we have \(\mathbf{r}'(\pi/4) = \langle \frac{3}{2\sqrt{2}}, -\frac{3}{2\sqrt{2}}, 1 \rangle\) and so
\[
\mathbf{l}(t) = \left< \frac{1}{2\sqrt{2}} + \frac{3t}{2\sqrt{2}}, \frac{1}{2\sqrt{2}} - \frac{3t}{2\sqrt{2}}, \frac{1}{2} + t \right>
\]

Answer: \(\left< \frac{1}{2\sqrt{2}} + 1 + 3t, 1 - 3t, \sqrt{2} + 2\sqrt{2}t \right>\)

(b) Compute the length of \(C\).

The length of \(C\) is:
\[
\int_0^{\pi/2} |\mathbf{r}'(\theta)| \, d\theta = \int_0^{\pi/2} \sqrt{9 \sin^4(\theta) \cos^2(\theta) + 9 \cos^4(\theta) \sin^2(\theta) + 4 \sin^2(\theta) \cos^2(\theta)} \, d\theta
\]
\[
= \int_0^{\pi/2} \sqrt{\frac{3}{2}} \sin(\theta) \cos(\theta) \, d\theta
\]
\[
= \frac{\sqrt{3}}{2} \bigg|_0^{\pi/2} \sin^2(\theta) = \frac{\sqrt{3}}{2}
\]

Answer: \(\frac{\sqrt{3}}{2}\)
Problem 3. \((5 + 5 + 5 + 5 = 20 \text{ points})\) Consider the function \(f(x, y) = y^2 - x^2 - xy^3\).

(a) Compute \(\nabla f(1, -1)\).

Answer: \(\nabla f(1, -1) = \langle 1, -3 \rangle\)

(b) Find an equation for the tangent plane to the graph of \(z = f(x, y)\) at the point \((1, -1, 1)\).

The tangent plane is perpendicular to the gradient of the equation \(0 = f(x, y) - z\) at the point \((1, -1, 1)\). Hence we use

\[\vec{n} = \langle 1, -3, -1 \rangle\]

Point: \((1, -1, 1)\) \(\rightarrow\) \(x - 3y - z - 3 = 0\)

Answer: \(x - 3y - z - 3 = 0\).
(c) Compute the directional derivative of $f$ at the point $(1, -1)$ in the direction of $(3, 1)$.

\[
(\nabla f) \langle 1, 1 \rangle = \nabla f \langle 1, 1 \rangle \cdot \frac{\langle 3, 1 \rangle}{\sqrt{10}} = 0.
\]

(d) True or false: The level curve defined by $f(x, y) = 1$ has, at the point $(1, -1)$, tangent line parallel to $(3, 1)$. Justify your answer.

Since $\nabla f \langle 1, -1 \rangle \cdot \langle 3, 1 \rangle = 0$, we conclude that $\langle 3, 1 \rangle$ is perpendicular to $\nabla f \langle 1, -1 \rangle$. Hence, as we are in 2-D, $\langle 3, 1 \rangle$ is parallel to the tangent to the level curve $f(x, y) = 1$ at the point $(1, -1)$.

Answer: (d) True
Problem 4. \((10+5+5 = 20\text{ points})\) Throughout this problem \(f(x, y) = x^2 + y^2 - xy.\)
(a) Let \(S^1\) denote the unit circle in the plane, that is, \(S^1 = \{(x, y) \mid x^2 + y^2 = 1\}\). Use the method of Lagrange multipliers to find the maximum and minimum values obtained by \(f(x, y)\) on \(S^1\).

Since we are restricting to the curve \(x^2 + y^2 = 1\), we have
\[
\begin{align*}
  f(x, y) &= 1 - xy \\
  g(x, y) &= x^2 + y^2 = 1.
\end{align*}
\]

Using Lagrange multipliers, we want to solve:
\[
\begin{align*}
  -y &= \lambda 2x \\
  -x &= \lambda 2y \\
  x^2 + y^2 &= 1.
\end{align*}
\]

Thus, \(\lambda = \frac{-y}{2x} = \frac{-x}{2y}\) \(\Rightarrow y^2 = x^2\). So
\[
2x^2 = 1 \Rightarrow x = \pm \frac{1}{\sqrt{2}}
\]
\[
y = \pm \frac{1}{\sqrt{2}}
\]

So points of interest: \(\frac{1}{2}\) \(\frac{3}{2}\)

\[
\begin{array}{c|c}
\text{value of } f & \\
(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & \frac{1}{2} \\
(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) & \frac{3}{2} \\
(-\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) & \frac{3}{2} \\
(-\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) & \frac{1}{2} \\
\end{array}
\]

Answer: \(\frac{1}{2}, \frac{3}{2}\)
(b) Find the critical point of \( f(x, y) \).

We want to solve

\[
\nabla f = 0
\]

\[
\iff
\]

\[
<2x-y, 2y-x> = <0, 0>
\]

\[
\Rightarrow y = 2x \quad x = 2y
\]

\[
\Rightarrow y = 4y \Rightarrow y = 0, x = 0
\]

Answer: (b) \((0,0)\)

(c) Let \( D \) denote the unit disk; that is, \( D = \{(x, y) \mid x^2 + y^2 \leq 1\} \). Combine parts (a) and (c) to determine the maximum and minimum values for \( f(x, y) \) on \( D \).

\[
\begin{array}{c|c}
(x, y) & f(x, y) \\
\hline
(0, 0) & 0 \\
(\sqrt{\frac{1}{2}}, \sqrt{\frac{1}{2}}) & \frac{1}{2} \\
(\sqrt{\frac{1}{2}}, -\sqrt{\frac{1}{2}}) & \frac{3}{2} \\
\end{array}
\]

Answer: (c) \(0, \frac{3}{2}\)
Problem 5. (5+7+8=20 points) Throughout this problem \( f(x, y) = x^3 + y^2 - 6xy + 6x + 3y. \)
(a) Compute \( \nabla f \) at an arbitrary point \((x, y)\).

Answer: \( (a) \nabla f(x, y) = \langle 3x^2 - 6y + 6, 2y - 6x + 3 \rangle \)

(b) Find all critical points of \( f(x, y) \).

\[
\nabla f(x, y) = 0 \\
\Rightarrow 2y = 6x - 3 \quad \text{and} \quad 3x^2 - 6y + 6 = 0 \\
\Rightarrow 3x^2 - 18x + 15 = 0 \\
\Rightarrow (3x - 3)(x - 5) = 0 \\
\Rightarrow x = 1 \quad \text{or} \quad x = 5
\]

Answer: \( (b) (1, \frac{3}{2}), (5, \frac{27}{2}) \)
(c) Use the second derivatives test to classify the critical points found in part (b).

\[ f_{xx}(x,y) = 6x \]
\[ f_{yy}(x,y) = 2 \]
\[ f_{xy} = -6 \]
\[ D = (6x)^2 - (-6)^2 = 12x - 36 \]
\[ = 12(x-3) \]

For \((1, 3/2)\): \(D < 0 \Rightarrow \) saddle point

For \((5, 2^{1/2})\): \(D > 0\) and \(f_{yy} > 0\) so local min.

Answer:

\[ D \]
\[ (1, 3/2) \): Saddle
\((5, 2^{1/2})\): local min here
Problem 6. (2x5=10 points) Let \( f(x, y) = e^{x-y} \).

(a) Suppose \( x \) and \( y \) are functions of \( t \) with \( x(3) = y(3) = 1/2 \) and \( x'(3) = 2 \) while \( y'(3) = 4 \). Compute \( \frac{df}{dt}(3) \).

\[
\frac{df}{dt} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}
\]

\[
\Rightarrow f_t(3) = f_x(1/2, 1/2) x'(3) + f_y(1/2, 1/2) y'(3)
\]

\[
= 1 \cdot 2 + (-1) \cdot 4 = \boxed{-2}
\]

Note: \( f_x = e^{x-y} \)

\( f_y = -e^{x-y} \)

Answer: \( -2 \)

(b) Suppose \( x \) and \( y \) are functions of \( s \) and \( t \) with \( x(3, 2) = 7, y(3, 2) = -5, x_s(3, 2) = 8, x_t(3, 2) = 1, y_s(3, 2) = 5 \) and \( y_t(3, 2) = -1 \). Compute the partial of \( f \) with respect to \( t \) at the point \( (3, 2) \) (that is, the point where \( s = 3 \) and \( t = 2 \)).

\[
\frac{f_t}{t} = f_x \frac{dx}{dt} + f_y \frac{dy}{dt}
\]

\[
\Rightarrow f_t(3, 2) = e^{7-(-5)}(1) + (-e^{7-(-5)})(-1)
\]

\[
= e^{12} - e^{12}(-1) = \boxed{2e^{12}}
\]

Answer: \( \boxed{2e^{12}} \)