Problem 1. (15 pts.) Consider the points \( A(1, -3, 2), B(3, -1, -1), C(-4, -9, 8), \) and \( D(-2, 4, 4). \)

(a – 5 pts.) Find an equation of the plane \( \Pi \) containing the points \( A, B, \) and \( C. \)

Solution. To write down an equation of the plane, we need to know a point on the plane (any of \( A, B, \) or \( C \) will do) and the normal vector. In this problem we can easily find two vectors in the plane, for example, \( \overrightarrow{AB} = (2, 2, -3) \) and \( \overrightarrow{AC} = (-5, -6, 6). \) Then their cross-product will be perpendicular to the plane:

\[
\overrightarrow{n} = \overrightarrow{AB} \times \overrightarrow{AC} = \begin{vmatrix}
  \mathbf{i} & \mathbf{j} & \mathbf{k} \\
  2 & 2 & -3 \\
  -5 & -6 & 6 \\
\end{vmatrix} = (12 - 18, 15 - 12, -12 + 10) = (-6, 3, -2)
\]

The equation of the plane then is

\[
\overrightarrow{n} \cdot \overrightarrow{PP_0} = 0 \quad \text{or} \quad \overrightarrow{n} \cdot \overrightarrow{OP} = \overrightarrow{n} \cdot \overrightarrow{OP_0},
\]

where \( P(x, y, z) \) is a generic point on the plane and \( P_0(x_0, y_0, z_0) \) is a reference point. We have:

\[
\begin{align*}
-6x + 3y - 2z &= -6(1) + 3(-3) - 2(2) & \text{taking } P_0 = A \\
-6x + 3y - 2z &= -6(3) + 3(-1) - 2(-1) & \text{taking } P_0 = B \\
-6x + 3y - 2z &= -6(-4) + 3(-9) - 2(8) & \text{taking } P_0 = C \\
&= -19
\end{align*}
\]

(b – 5 pts.) Find the area of \( \triangle ABC. \)

Solution. The area of \( \triangle ABC \) is half the area of the parallelogram spanned by the vectors \( \overrightarrow{AB} \) and \( \overrightarrow{AC}, \) so it can be computed with the help of the cross-product:

\[
\text{Area}(\triangle ABC) = \frac{1}{2} \text{Area}(\overrightarrow{AB}, \overrightarrow{AC}) = \frac{1}{2} |\overrightarrow{AB} \times \overrightarrow{AC}| = \frac{1}{2} \sqrt{(-6)^2 + (3)^2 + (-2)^2} = \frac{1}{2} \sqrt{49} = \frac{7}{2}.
\]

(c – 5 pts.) Find the distance from the point \( D \) to the plane \( \Pi \) containing the points \( A, B, \) and \( C. \)

Solution. Consider a vector from any point in the plane (we will use \( A \)) to the point \( D, \overrightarrow{AD} = (-3, 7, 2). \) Then \( d(D, \Pi) \) is just the (absolute value of) the component of this vector in the direction normal to the plane:

\[
d(D, \Pi) = |\text{comp}_n \overrightarrow{AD}| = \left| \overrightarrow{AD} \cdot \frac{\overrightarrow{n}}{|\overrightarrow{n}|} \right| = \frac{|(-3, 7, 2) \cdot (-6, 3, -2)|}{7} = \frac{35}{7} = 5.
\]

(c’ – 5 pts.) Find the volume of the parallelepiped spanned by the vectors \( \overrightarrow{AB}, \overrightarrow{AC}, \) and \( \overrightarrow{AD}. \)

Solution.

\[
\begin{align*}
\text{Volume}(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) &= |\text{det}(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD})| \\
&= \begin{vmatrix}
  -3 & 7 & 2 \\
  2 & 2 & -3 \\
  -5 & -6 & 6 \\
\end{vmatrix} = |(-3, 7, 2) \cdot (-6, 3, -2)| = 35.
\end{align*}
\]

Note that

\[
\text{Volume}(\overrightarrow{AB}, \overrightarrow{AC}, \overrightarrow{AD}) = \text{Area(Base)} \times \text{Height} = \text{Area}(\overrightarrow{AB}, \overrightarrow{AC}) \times d(D, \Pi) = 7 \times 5.
\]

Problem 2. (10 pts.)

(a – 5 pts.) Find parametric equations of the line of intersection of the planes \( x - 3y + 2z = -1 \) and \( 4x + y + 7z = 9.\)
Solution. To write down a parametric equation of a line, we need a point on the line and a direction vector. Since our line is the line of intersection of two planes, its direction vector should be perpendicular to normal vectors \( \mathbf{n}_1 = (1, -3, 2) \) and \( \mathbf{n}_2 = (4, 1, 7) \) of the planes. We use the cross-product again:

\[
\mathbf{v} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -3 & 2 \\ 4 & 1 & 7 \end{vmatrix} = (-23, 1, 13).
\]

To find a point on the line, we set one of the coordinates to zero and solve for the other two. For example, setting \( z = 0 \), we get

\[
\begin{align*}
    x - 3y &= -1, \\
    4x + y &= 9,
\end{align*}
\]

and we see that the point \( P(2, 1, 0) \) is on both planes. So we get the following parametric equations:

\[
\mathbf{r}(t) = \mathbf{OP} + t\mathbf{v} = \langle 2, 1, 0 \rangle + t\langle -23, 1, 13 \rangle.
\]

or, in components,

\[
\begin{align*}
    x(t) &= 2 - 23t, \\
    y(t) &= 1 + t, \\
    z(t) &= 13t.
\end{align*}
\]

(b – 5 pts.) Find parametric equations of the line through the points \( A(3, -1, 2) \) and \( B(5, 1, 3) \). What is the interval of change of the parameter that you need to parametrize just the line segment \( AB \)?

Solution. Using the vector \( \mathbf{AB} = (2, 2, 1) \) as a direction vector, and the point \( A \) as the reference point, we get the following parametric equations:

\[
\mathbf{r}(t) = \mathbf{OA} + t\mathbf{AB} = (3, -1, 2) + t(2, 2, 1)
\]

or, in components,

\[
\begin{align*}
    x(t) &= 3 + 2t, \\
    y(t) &= -1 + 2t, \\
    z(t) &= 2 + t.
\end{align*}
\]

Note that for this parametrization \( \mathbf{r}(0) = \mathbf{OA} \) and \( \mathbf{r}(1) = \mathbf{OB} \), so in order to parametrize just the line segment \( AB \) we take \( 0 \leq t \leq 1 \).

Problem 3. (10 pts.)

(a – 5 pts.) Find parametric equations of the tangent line to the parametric curve \( \mathbf{r}(t) = (2 + t^3, 1 - 4t, 5 - t^2) \) at the time \( t = 1 \).

Solution. At time \( t = 1 \) we have

\[
\begin{align*}
    \mathbf{r}(1) &= (3, -3, 4) \quad \text{(reference point)} \\
    \mathbf{r}'(1) &= (3t^2, -4, -2t) \bigg|_{t=1} = (3, -4, -2) \quad \text{(direction vector)}
\end{align*}
\]

and we get the following parametric equations:

\[
\mathbf{r}(s) = (3, -3, 4) + s(3, -4, -2)
\]

(b – 5 pts.) Find the point of intersection of this line with the xy-plane.

Solution.

Rewriting the above equations in components,

\[
\begin{align*}
    x(s) &= 3 + 3t, \\
    y(s) &= -3 - 4t, \\
    z(s) &= 4 - 2t
\end{align*}
\]

and setting \( z = 0 \), we see that this line intersects the xy-plane when \( s = 2 \). Evaluating \( x \) and \( y \) at \( s = 2 \), \( x(2) = 9, y(2) = -11 \), we get the point of intersection \( Q(9, -11, 0) \).
**Problem 4. (5 pts.)**

Use the level curves of the function \( f(x, y) \) on the graph on the right to approximate \( f_x(-1, 1), f_y(-1, 1), f_x(1, 1), f_y(1, 1) \).

**Solution.** First consider the point \((-1, 1)\). Looking at the intersection of the horizontal line \( y = 1 \) with the level curves we see that to the left and to the right of the point \((-1, 1)\) it intersects with the same level \( f(x, y) = 13 \), so \( f_x(-1, 1) \approx 0 \). In the vertical direction two nearby levels are \( f(x, y) = 13 \) and \( f(x, y) = 10 \). Since the spacing between these levels is \( \approx 5 \), \( f_y(-1, 1) \approx \frac{10 - 13}{0.5} = -6 \). Looking at the level curves near \((1, 1)\) we see that it seems to be a local maximum, so we expect both \( f_x(1, 1) \) and \( f_y(1, 1) \) to vanish.

The exact values for these partial derivatives turn out to be equal to our estimates: \( f_x(-1, 1) = 0, f_y(-1, 1) = -6, f_x(1, 1) = 0, \) and \( f_y(1, 1) = 0 \).

**Problem 5. (10 pts.)** A particle moves so that \( \mathbf{r}(t) = (3 \cos t + 3t \sin t) \mathbf{i} + (3 \sin t - 3t \cos t) \mathbf{j} + (2t^2) \mathbf{k} \). (a - 5 pts.) Find the distance travelled by the particle from \( t = 0 \) to \( t = 4\pi \).

**Solution.** We have:

\[
\mathbf{r}'(t) = (3 \cos t, 3t \sin t, 4t)
\]

(velocity)

\[
s(t) = |\mathbf{r}'(t)| = \sqrt{9 \cos^2 t + 9t^2 \sin^2 t + 16t^2}
= \sqrt{9t^2 + 16t^2} = 5t \quad \text{(since } t \geq 0) \quad \text{(speed)}
\]

The distance travelled is just the integral of speed w.r.t. time (the arclength formula):

\[
\text{Arclength} = \int_0^{4\pi} 5t \, dt = \frac{5}{2} t^2 \bigg|_0^{4\pi} = 40\pi^2 \approx 394.78
\]

(b - 5 pts.) Find the distance between \( \mathbf{r}(0) \) and \( \mathbf{r}(4\pi) \).

**Solution.** Computing \( \mathbf{r}(0), \mathbf{r}(4\pi), \) and using the distance formula we get:

\[
\mathbf{r}(0) = (3, 0, 0), \quad \mathbf{r}(4\pi) = (3, -12\pi, 32\pi^2)
\]

\[
d(\mathbf{r}(0), \mathbf{r}(4\pi)) = \sqrt{(3 - 3)^2 + (-12\pi - 0)^2 + (32\pi^2 - 0)^2} = \sqrt{4\pi^2(9 + 64\pi^2)}
= 4\pi \sqrt{9 + 64\pi^2} \approx 318.07.
\]

**Problem 6. (15 pts.)** Consider the function \( f(x, y) = x^2 e^{xy} \) and the point \( P(3, 0, 9) \). (a - 5pts.) Find an equation of the tangent plane to the graph of \( z = f(x, y) \) at the point \( P \).

**Solution.** Using the equation of the tangent plane,

\[
z = f(3, 0) + f_x(3, 0)(x - 3) + f_y(3, 0)(y - 0)
\]

and evaluating the corresponding partial derivatives,

\[
f_x(x, y) = 2xe^{xy} + x^2 ye^{xy} \quad \quad f_y(x, y) = x^2 e^{xy} \quad \quad f_x(3, 0) = 6 \quad \quad f_y(3, 0) = 27
\]

we get

\[
z = 9 + 6(x - 3) + 27y
\]
(b – 5pts.) Find parametric equations of the normal line to the graph of \( z = f(x, y) \) going through the point \( P \).

**Solution.** The direction vector of the normal line is just the normal vector to the tangent plane that we can obtain either from the equation in part (a) (it may help to first rewrite it as \(-6x - 27y + z = -18\)) or by using the cross-product of two tangent coordinate vectors for a parametric surface \( r(x, y) = \langle x, y, f(x, y) \rangle \):

\[
\mathbf{n} = \mathbf{r}_x(3, 0) \times \mathbf{r}_y(3, 0) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 0 & f_x(3, 0) \\ 0 & 1 & f_y(3, 0) \end{vmatrix} = (-f_x(3, 0), -f_y(3, 0), 1) = (-6, -27, 1)
\]

or by representing the graph as a level surface \( F(x, y, z) = z - f(x, y) = 0 \) and computing

\[
\mathbf{n} = \nabla F(P) = (-f_x(3, 0), -f_y(3, 0), 1) = (-6, -27, 1).
\]

Then the parametric equations of the normal line are

\[
\mathbf{r}(t) = \mathbf{OP} + t\mathbf{n} = \langle 3, 0, 9 \rangle + t\langle -6, -27, 1 \rangle = \langle 3 - 6t, -27t, 9 + t \rangle.
\]

\[\clubsuit\]

(c – 5pts.) Use differentials to approximate \( f(2.9, 0.2) \).

**Solution.** Using differentials to approximate \( f(2.9, 0.2) \) is the same as evaluating it using the linearization (or the right-hand-side of the tangent plane equation in part (a)):

\[
f(2.9, 0.2) \approx 9 + 6(2.9 - 3) + 27(0.2 - 0) = 13.8
\]

(which is of course the same as computing

\[
f(2.9, 0.2) = f(3, 0) + \Delta z \approx f(3, 0) + f_x(3, 0)\Delta x + f_y(3, 0)\Delta y
\]

\[\diamondsuit\]

**Problem 7.** (15 pts.) The capacity, \( C \), of a communication channel, such as a telephone line, to carry information depends on the ratio of the signal strength, \( S \), to the noise, \( N \) in the following way

\[
C = k \ln \left( 1 + \frac{S}{N} \right),
\]

where \( k > 0 \) is some constant. Suppose that the signal and noise are given as a function of time, \( t \) in seconds, by

\[
S(t) = 4 + \cos(4\pi t) \quad N(t) = 2 + \sin(2\pi t).
\]

What is \( \frac{dC}{dt} \) one second after the transmission started? Is the capacity increasing or decreasing at that instant?

**Solution.** Using the Chain Rule, we get:

\[
C(S, N) = k \ln \left( 1 + \frac{S}{N} \right) \quad C(5, 2) = k \ln \left( \frac{5}{2} \right)
\]

\[
\begin{align*}
C_S &= \frac{k}{1 + \frac{S}{N}} & C_N &= -k \frac{S}{N(S+N)} & C_S(5,2) &= \frac{k}{5} & C_N(5,2) &= -k \frac{5}{7} \\
S(t) &= 4 + \cos(4\pi t) & N(t) &= 2 + \sin(2\pi t) & S(1) &= 5 & N(1) &= 2 \\
S_t &= -4\pi \sin(4\pi t) & N_t &= 2\pi \cos(2\pi t) & S_t(1) &= 0 & N_t(1) &= 2\pi
\end{align*}
\]

and so

\[
\frac{dC}{dt} (1) = \frac{\partial C}{\partial S} (5, 2) \frac{dS}{dt} (1) + \frac{\partial C}{\partial N} (5, 2) \frac{dN}{dt} (1) = -\frac{5\pi k}{7} < 0,
\]

since \( k > 0 \), so the capacity is decreasing. Below is the graph of \( S(t) \), \( C(t) \), and \( N(t) \) with \( k = 1 \):
Problem 8. (10 pts.) True or False?

1. The cross-product of two unit vectors is a unit vector.

Solution. False. The magnitude of the cross-product of two vectors depends not only of their magnitudes, but also on the angle between them ($|u \times v| = |u||v|\sin(\theta)$); it will be unit in the vectors are orthogonal to each other, but will be smaller otherwise (for an extreme example, take $i \times i = 0$.) ♣

2. If $u, v,$ and $w$ are all non-zero vectors in space and $u \cdot v = u \cdot w$, then $v = w$.

Solution. False. The fact that $v$ and $w$ have the same components in the $u$ direction does not make them equal. For example, take $u = i = (1, 0, 0)$, $v = (2, 1, 1)$, $w = (2, -3, 5)$. ♣

3. The value of $u \cdot (u \times v)$ is always zero.

Solution. True. Up to a sign, $u \cdot (u \times v)$ is equal to the volume of the parallelepiped determined by the vectors $u, u,$ and $v$, which is zero, since two of the vectors are the same. ♣

4. The value of $u \times (u \times v)$ is always zero.

Solution. False. For example, $i \times (i \times j) = i \times k = -j$. ♣

5. If two planes in space do not intersect, their normal vectors are proportional.

Solution. True. If two planes in space do not intersect, they are parallel, and so their normal vectors are proportional. ♣

6. If two lines in space do not intersect, their direction vectors are proportional.

Solution. False. The lines can be skew. ♣

7. If a particle moves along the circle, then its acceleration vector is always perpendicular to its velocity vector.

Solution. False. If the motion is non-uniform, acceleration will have a tangential component. ♣

8. If a particle moves with constant speed, the path of the particle must be a line.

Solution. False. Particle moving with constant speed can be moving along curves (in fact, any curve parametrized w.r.t. the arclength gives an example of such motion). On the other hand, particles moving with constant velocity indeed move along straight lines. ♣
If you know the gradient vector $\nabla f(P)$, then you can find the directional derivative $D_{u}f(P)$ for any unit vector $u$.

Solution. True. $D_{u}f(P) = \nabla f(P) \cdot u$. ♣

If you know the directional derivative $D_{u}f(P)$ for any unit vector $u$, then you can find the gradient vector $\nabla f(P)$.

Solution. True. Just pick the direction $u_{max}$ that maximizes $D_{u}f(P)$, then $\nabla f(P) = (D_{u_{max}}f(P))u_{max}$. ♣

Problem 9. (10 pts.) Match the following functions, graphs, level curves.

Solution. (a) $f(x, y) = x^3 y^3 e^{-x^2-y^2}$. The exponential term makes this function decay rapidly away from the origin. Also, it is symmetric w.r.t. change $x \leftrightarrow y$, but antisymmetric w.r.t. change $x \leftrightarrow -x$ and $y \leftrightarrow -y$.

(b) $f(x, y) = \frac{-10y}{x^2 + y^2 + 1}$. The denominator makes this function decay away from the origin as well, but the decay is much slower than in the exponential case. Also, this function is antisymmetric w.r.t. change $y \leftrightarrow -y$ and vanishes on the $x$-axes.

(c) $f(x, y) = y^2 \sin(x)$. This function is a sine wave in $x$ whose amplitude is given by $y^2$.

(d) $f(x, y) = e^{-x^2} + e^{-y^2}$. This function is again decaying exponentially, but in this case it is away from the coordinate axes. Also, it is always positive and is symmetric w.r.t. changes $x \leftrightarrow y$, $x \leftrightarrow -x$, and $y \leftrightarrow -y$. 
(e) $f(x, y) = \cos^2 x + y$. This function is a wave in the $x$-direction and has a steady linear growth in the $y$-direction. Also, its level curves are given by the equation $y + \cos^2 x = k$ or $y = k - \cos^2 x$, so they look like a collection of $\cos^2 x$-waves shifted in the $y$-direction.

(IV)  

(B)