1.4) a) Since $1=1$, $1+3=4$, $1+3+5=9$ and $1+3+5+7=16$, one might naturally conjecture that $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

b) Claim: $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

Proof: For all $n \in \mathbb{N}$, let $P_n$ be the mathematical statement

\[ “1 + 3 + 5 + \cdots + (2n-1) = n^2.” \]

$P_1$ is clearly true, since $1 = 1^2$.

Suppose that $P_n$ is true for some $n \in \mathbb{N}$. Then, $1 + 3 + 5 + \cdots + (2n-1) = n^2$. But, one may then apply our inductive assumption to check that

\[ 1 + 3 + 5 + \cdots + (2n-1) + (2(n+1)-1) = n^2 + (2n+1) = (n+1)^2. \]

So, $P_{n+1}$ is true. Thus, $P_{n+1}$ is true whenever $P_n$ is true.

Therefore, since $P_1$ is true and $P_{n+1}$ is true whenever $P_n$ is true, the principle of mathematical induction implies that $P_n$ is true for all $n \in \mathbb{N}$. So, $1 + 3 + 5 + \cdots + (2n-1) = n^2$ for all $n \in \mathbb{N}$.

1.5) Claim: $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

Proof: For all $n \in \mathbb{N}$, let $P_n$ be the mathematical statement

\[ “1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}.” \]

$P_1$ is true since $1 + \frac{1}{2} = 2 - \frac{1}{2} = \frac{3}{2}$.

Suppose $P_n$ is true, i.e. that $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$. Then, we may apply our inductive hypothesis to see that:

\[ 1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^n} + \frac{1}{2^{n+1}} = 2 - \frac{1}{2^{n+1}}. \]

So, $P_{n+1}$ is true. Thus, $P_{n+1}$ is true whenever $P_n$ is true.

Therefore, since $P_1$ is true and $P_{n+1}$ is true whenever $P_n$ is true, $P_n$ is true for all $n \in \mathbb{N}$, by the principle of mathematical induction. So, $1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^n} = 2 - \frac{1}{2^n}$ for all $n \in \mathbb{N}$.

1.7) Claim: $7^n - 6n - 1$ is divisible by $36$ for all $n \in \mathbb{N}$.

Proof: For all $n \in \mathbb{N}$, let $P_n$ be the mathematical statement

\[ “7^n - 6n - 1$ is divisible by $36.” \]

Then, $P_1$ is true, since $7^1 - 6(1) - 1 = 0$ which is divisible by $36$, since $0 = 36(0)$.

Suppose that $P_n$ is true for some $n \in \mathbb{N}$, i.e. $7^n - 6n - 1$ is divisible by $36$. So, there exists $m \in \mathbb{Z}$ so that $7^n - 6n - 1 = 36m$. Then,

\[ 7^{n+1} - 6(n+1) - 1 = 7(7^n - 6n - 1) + 36n = 7(36m) + 36n = 36(7m + n). \]

Since $7n + m \in \mathbb{Z}$ and $7^{n+1} - 6(n+1) - 1 = 36(7m + n)$, $7^{n+1} - 6(n+1) - 1$ is divisible by $36$, so $P_{n+1}$ is true.
Therefore, since \( P_1 \) is true and \( P_{n+1} \) is true whenever \( P_n \) is true, the principle of mathematical induction implies that \( P_n \) is true for all \( n \in \mathbb{N} \), so \( 7^n - 6n - 1 \) is divisible by 36 for all \( n \in \mathbb{N} \).

1.11) For all \( n \in \mathbb{N} \), let \( P_n \) be the mathematical statement

\[
“n^2 + 5n + 1” \text{ is an even integer.”}
\]

a) Claim: If \( P_n \) is true for some \( n \in \mathbb{N} \), then \( P_{n+1} \) is true.
Proof: If \( P_n \) is true, then \( n^2 + 5n + 1 \) is an even integer. So

\[
(n + 1)^2 + 5(n + 1) + 1 = (n^2 + 5n + 1) + 2n + 6
\]

is an even integer, since it is a sum of 3 even integers. (Notice that \( n^2 + 5n + 1 \) is even by assumption, and \( 2n \) and 6 are even since they are multiples of 2.) Therefore, \( P_{n+1} \) is true.

b) \( P_n \) is never true, which illustrates the importance of checking the base case when attempting to apply the principle of mathematical induction.

Proof that \( P_n \) is never true: We divide into the case where \( n \) is even and where \( n \) is odd. If \( n \) is even, then \( n^2 + 5n + 1 \) is odd, since it is the sum of two even integers, \( n^2 \) and \( 5n \), and the odd integer 1. If \( n \) is odd, then \( n^2 + 5n + 1 \) is odd, since it is the sum of three odd integers, \( n^2 \), \( 5n \) and 1. (This last claim uses the fact that a product of odd integer is odd.) Therefore, \( n^2 + 5n + 1 \) is never an even integer if \( n \in \mathbb{N} \), so \( P_n \) is never true.

2.4) Claim: \((5 - \sqrt{3})^{\frac{2}{3}} \) is an irrational number.

Proof: One may check that \( z = (5 - \sqrt{3})^{\frac{2}{3}} \) is a root of the polynomial \( x^6 - 10x^3 + 22 \) which has integral coefficients.

Suppose that \( z \) is a rational number. One may then write \( z = \frac{p}{q} \) where \( p \) and \( q \) are integers without common prime factors. The Rational Zeros Theorem then implies that \( q \) divides 1 and \( p \) divides 22. Then, \( z \) must be either 1, \(-1\), 2, \(-2\), 11, \(-11\), 22 and \(-22\). However, one may check by inspection that none of these are roots of \( x^6 - 10x^3 + 22 \) (since \( 1^6 - 10(1^3) + 22 = 13 \), \((-1)^6 - 10((-1)^3) + 22 = 33 \), \( 2^6 - 10(2^3) + 22 = 6 \), \( 2^6 - 10((-2)^3) + 22 = 166 \), \( 11^6 - 10((11)^3) + 22 = 1,758,273 \), \((-11)^6 - 10((-11)^3) + 22 = 1,784,893 \), \( 22^6 - 10((22)^3) + 22 = 113,273,446 \), and \( 22^6 - 10((22)^3)^3 + 22 = 113,486,406 \). Therefore, \( z \) cannot be any of these numbers. This contradiction establishes that \( z \) must be an irrational number.

4.6) Let \( S \) be a non-empty bounded subset of \( \mathbb{R} \).

a) Claim: \( \inf S \leq \sup S \).

Proof: First notice that since \( S \) is bounded, the completeness axiom and corollary 4.5 guarantee that \( \sup S \) and \( \inf S \) exist.

Since \( S \) is non-empty, there exists \( s_0 \in S \). Since \( \inf S \) is a lower bound for \( S \), \( \inf S \leq s_0 \). Since \( \sup S \) is an upper bound for \( S \), \( s_0 \leq \sup S \). Thus, \( \inf S \leq s_0 \leq \sup S \), so \( \inf S \leq \sup S \).

b) Claim: If \( \inf S = \sup S \), then \( S = \{ \sup S \} \).

Proof: If \( s \in S \), then, as in part a), \( \inf S \leq s \leq \sup S \). Since, \( \inf S = \sup S \), this implies that \( s = \sup S \). Since \( S \) is non-empty, \( S = \{ \sup S \} \).

4.8) Suppose that \( S \) and \( T \) are non-empty subsets of \( \mathbb{R} \) and that \( s \leq t \) whenever \( s \in S \) and \( t \in T \).

a) Claim: \( S \) is bounded above and \( T \) is bounded below.

Proof: If \( t_0 \) is any element of \( T \), then \( s \leq t_0 \) for all \( s \in S \), so \( t_0 \) is an upper bound for \( S \). Hence, \( S \) is bounded above.

Similarly, if \( s_0 \) is any element of \( S \), then \( s_0 \leq t \) for all \( t \in T \), so \( s_0 \) is a lower bound for \( T \). Hence, \( T \) is bounded below.
b) Claim: sup $S \leq \inf T$.

Proof: In part a) we observed that if $t \in T$, then $t$ is an upper bound for $S$. Since sup $S$ is the least upper bound for $S$, $t \geq \sup S$. Since sup $S \leq t$ for all $t \in T$, sup $S$ is a lower bound for $T$. Since inf $T$ is the greatest lower bound for $T$, sup $S \leq \inf T$.

c) Example: Let $S = [1, 2]$ and $T = [2, 3]$. Then sup $S = \inf T = 2$, $S \cap T = \{2\}$ is non-empty.

d) Example: Let $S = [1, 2)$ and $T = (2, 3]$. Then sup $S = \inf T = 2$, yet $S \cap T$ is empty.

4.12) Claim: If $a, b \in \mathbb{R}$ and $a < b$, then there exists an irrational number $x$ such that $a < x < b$.

Proof: We first establish the lemma suggested by the hint:

Lemma: If $r$ is rational, then $r + \sqrt{2}$ is irrational.

Proof of lemma: Suppose that there exists a rational number $r$ such that $r + \sqrt{2}$ is also rational. Then, $\sqrt{2} = (r + \sqrt{2}) - r$ is the difference of two rational numbers. However, it is easily checked that the difference of two rational numbers is also a rational number. (If $s, t \in \mathbb{Q}$, then $s = \frac{m}{n}$ and $t = \frac{n}{q}$, where $m, n, p, q \in \mathbb{Z}$ and $n$ and $q$ are non-zero. So, $s - t = \frac{mq - np}{nq}$ is clearly rational.) Therefore, $\sqrt{2}$ must be rational. However, we have previously shown that $\sqrt{2}$ is irrational. Having achieved a contradiction, we see that it must be the case that $r + \sqrt{2}$ is irrational. This completes the proof of the lemma.

Now notice that since $a < b$, it is also the case that $a - \sqrt{2} < b - \sqrt{2}$. The density of rational numbers, result 4.7, then implies that there exists a rational number $r$ such that $a - \sqrt{2} < r < b - \sqrt{2}$. But then $a < r + \sqrt{2} < b$ and, by the lemma, $r + \sqrt{2}$ is an irrational number. This establishes our claim.