32.1) Claim: \( f(x) = x^3 \) is integrable on \([0, b]\) for all \( b > 0\), and \( U(f) = L(f) = \int_0^b f = \frac{b^4}{4} \).

Proof: Let \( P = \{0 = t_0 < t_1 \cdots < t_n = b\} \) be any partition of \([0, b]\). Then, since \( f \) is increasing, \( M(f, [t_{k-1}, t_k]) = t_k^3 \) and \( m(f, [t_{k-1}, t_k]) = (t_{k-1})^3 \) for all \( k \). So,

\[
U(f, P) = \sum_{k=1}^{n} t_k^3(t_k - t_{k-1})
\]

and

\[
L(f, P) = \sum_{k=1}^{n} t_{k-1}^3(t_k - t_{k-1}).
\]

For any \( n \in \mathbb{N} \) we can consider the specific partition \( P_n = \{0 = t_0 < t_1 \cdots < t_n = b\} \) where \( t_k = \frac{bk}{n} \) for all \( k \). In this case

\[
U(f, P_n) = \sum_{k=1}^{n} \left( \frac{kb}{n} \right)^3 \left( \frac{b}{n} \right) = \frac{b^4}{n^4} \sum_{k=1}^{n} k^3.
\]

Exercise 1.3 and Example 1 in section 1 give that \( \sum_{k=1}^{n} k^3 = \frac{1}{4} \frac{n^2(n+1)^2}{2} \), so

\[
U(f, P_n) = \frac{b^4}{4n^4} \left( \frac{n(n+1)^2}{2} \right) = \frac{b^4(n+1)^2}{8n^2}.\]

We have previously observed that \( \lim \frac{n^2}{n^2} = \lim 1 + \frac{1}{n} = 1 \), so

\[
U(f, P_n) = \frac{b^4}{8} \frac{(n+1)^2}{n^2} \leq \frac{b^4}{8} \frac{(n+1)^2}{n^2} \leq \frac{b^4}{8}.\]

Similarly,

\[
L(f, P_n) = \frac{b^4}{8} \frac{(n+1)^2}{n^2} \leq \frac{b^4}{8} \frac{(n+1)^2}{n^2} \leq \frac{b^4}{8} \frac{(n+1)^2}{n^2} \leq \frac{b^4}{8}.
\]

Since \( U(f) \leq U(f, P_n) \) for all \( n \), this implies that \( U(f) \leq \frac{b^4}{8} \).

32.6) Let \( f \) be a bounded function on \([a, b]\).

Claim: If there exist sequences \( (P_n) \) and \( (Q_n) \) of partitions of \([a, b]\) such that \( \lim U(f, P_n) - L(f, Q_n) = 0 \),
then \( f \) is integrable on \([a, b]\) and \( \int_a^b f = \lim U(f, P_n) = \lim L(f, Q_n) \).

Proof: Let \( \epsilon > 0 \). Then, since \( \lim U(f, P_n) - L(f, Q_n) = 0 \), there exists \( N \) such that \( n > N \)
implies that \( |U(f, P_n) - L(f, Q_n)| < \epsilon \). Choose \( m \in \mathbb{N} \) such that \( m > N \) and let \( P = P_m \cup Q_m \).

Then, Lemma 32.2 implies that

\[
L(f, Q_m) \leq L(f, P) \leq U(f, P) \leq U(f, P_m) < L(f, Q_m) + \epsilon.
\]

Therefore, \( U(f, P) - L(f, P) < \epsilon \).

So, for all \( \epsilon > 0 \), there exists a partition \( P \) of \([a, b]\) such that that \( U(f, P) - L(f, P) < \epsilon \).

The Cauchy criterion for integrals, Theorem 32.5, then guarantees that \( f \) is integrable on \([a, b]\). Since

\[
U(f) \leq U(f, P_n) < L(f, Q_n) + (U(f, P_n) - L(f, Q_n)) \leq L(f) + (U(f, P_n) - L(f, Q_n)) \leq U(f) + (U(f, P_n) - L(f, Q_n))
\]

and \( \lim(U(f, P_n) - L(f, Q_n)) = 0 \), the squeeze principle implies that \( U_n = (U(f, P_n)) \) converges to \( U(f) \), so \( \int_a^b f = \lim U_n \). Similarly, \( \int_a^b f = \lim L_n \).
33.4) Claim: There exists a function \( g \) which is not integrable, but \( |g| \) is integrable.

Example: Let \( g : [0,1] \to \mathbb{R} \) be defined by \( g(x) = .5 \) if \( x \) is rational and \( g(x) = -.5 \) if \( x \) is irrational. Then \( |g(x)| = .5 \) for all \( x \in [0,1] \). Hence, \( |g| \) is integrable, since constant functions are continuous, they are also integrable (by Theorem 33.2).

Suppose \( g \) were integrable, then the function \( f : [0,1] \to \mathbb{R} \) defined by \( f(x) = g(x) + .5 \) would also be integrable, by Theorem 33.3. But, \( f \) is exactly the function discussed in Example 2 of section 32, where it is established that \( f \) is not integrable. Therefore, \( g \) must also not be integrable.

33.7) Suppose that \( |f(x)| \leq B \) for all \( x \in [a,b] \).

a) Claim: If \( P \) is any partition of \([a,b]\), then

\[
U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)).
\]

Proof: Let \( P = \{0 = t_0 < t_1 \cdots < t_n = b\} \) be a partition of \([a,b]\). If \( x_0, y_0 \in [t_{k-1}, t_k] \), then

\[
f^2(x_0) - f^2(y_0) = (f(x_0) + f(y_0))(f(x_0) - f(y_0)) \leq |f(x_0) + f(y_0)| |f(x_0) - f(y_0)| \leq 2B|f(x_0) - f(y_0)| \leq 2B(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))
\]

Given any \( \epsilon > 0 \) there exists \( x_0, y_0 \in [t_{k-1}, t_k] \) such that \( f^2(x_0) > M(f^2, [t_{k-1}, t_k]) - \frac{\epsilon}{2} \) and \( f^2(y_0) < m(f^2, [t_{k-1}, t_k]) + \frac{\epsilon}{2} \). So,

\[
M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq f^2(x_0) - f^2(y_0) + \epsilon \leq 2B(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])) + \epsilon
\]

But, since the above inequality holds for all \( \epsilon > 0 \), one concludes that

\[
M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \leq 2B(M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k])).
\]

Therefore, by making use of the above inequality for each \( k \), we conclude that

\[
U(f^2, P) - L(f^2, P) = \sum_{k=1}^{n} \left( M(f^2, [t_{k-1}, t_k]) - m(f^2, [t_{k-1}, t_k]) \right) (t_k - t_{k-1}) \leq \sum_{k=1}^{n} 2B \left( M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]) \right) (t_k - t_{k-1}) \leq 2B \sum_{k=1}^{n} M(f, [t_{k-1}, t_k]) - m(f, [t_{k-1}, t_k]))(t_k - t_{k-1}) = 2B(U(f, P) - L(f, P)).
\]

Thus, \( U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) \) as claimed.

b) If \( f \) is integrable on \([a,b]\), then \( f^2 \) is integrable on \([a,b]\).

Proof: Given any \( \epsilon > 0 \), since \( f \) is integrable we may apply Theorem 32.5 to see that there exists a partition \( P \) of \([a,b]\), such that

\[
U(f, P) - L(f, P) < \frac{\epsilon}{2B}.
\]


From part a) we can then conclude, that

\[ U(f^2, P) - L(f^2, P) \leq 2B(U(f, P) - L(f, P)) < 2B \left( \frac{\epsilon}{2B} \right) = \epsilon. \]

Thus, for any \( \epsilon > 0 \), there exists a partition \( P \) of \([a,b]\), such that
\[ U(f^2, P) - L(f^2, P) < \epsilon. \] Theorem 32.5 then implies that \( f^2 \) is integrable on \([a,b]\).

33.8a) Claim: If \( f \) and \( g \) are integrable on \([a,b]\), then \( fg \) is integrable on \([a,b]\).
Proof: Suppose that \( f \) and \( g \) are integrable on \([a,b]\). Theorem 33.3(ii) implies that \( f + g \) and \( f - g = f + (-g) \) are integrable on \([a,b]\). Problem 33.7 implies that \((f + g)^2 \) and \((f - g)^2 \) are integrable on \([a,b]\). Theorem 33.3(i) implies that \(-f^2 \) is integrable on \([a,b]\). Theorem 33.3(ii) implies that \(4fg = (f + g)^2 + (-f - g)^2 \) is integrable on \([a,b]\). Finally, Theorem 33.3(i) implies that \(fg = \frac{1}{2}(f + g)^2 \) is integrable on \([a,b]\).

33.13) Claim: Suppose that \( f \) and \( g \) are continuous functions on \([a,b]\) and that \( \int_a^b f = \int_a^b g \). Then there exists \( x \in [a,b] \) such that \( f(x) = g(x) \).
Proof: Consider the function \( h = f - g \). Then, by Theorems 17.3 and 17.4, \( h \) is also continuous on \([a,b]\). Since \( f \), \( g \), and \( h \) are continuous on \([a,b]\), they are all integrable on \([a,b]\), by Theorem 33.2. Theorem 33.3 implies that \( \int_a^b h = \int_a^b f - \int_a^b g = 0 \). Theorem 33.9 then implies that there exists \( x \in [a,b] \) such that \( h(x) = \frac{1}{b-a} \int_a^b h = 0 \). Since \( h(x) = f(x) - g(x) \), this implies that \( f(x) = g(x) \).

34.11) Suppose that \( f \) is continuous on \([a,b]\).
Claim: If \( \int_a^b f(x)^2 dx = 0 \), then \( f(x) = 0 \) for all \( x \in [a,b] \).
Proof: Since \( f \) is continuous on \([a,b]\), Theorem 17.4 (part ii) implies that \( f^2 = ff \) is continuous on \([a,b]\). Theorem 33.4(part ii) then implies that \( f(x)^2 = 0 \) for all \( x \in [a,b] \). So, \( f(x) = 0 \) for all \( x \in [a,b] \).

34.12) Suppose that \( f \) is continuous on \([a,b]\).
Claim: If \( \int_a^b f(x)g(x) = 0 \) for any function \( g \) which is continuous on \([a,b]\), then \( f(x) = 0 \) for all \( x \in [a,b] \).
Proof: If \( \int_a^b f(x)g(x) = 0 \) for any function \( g \) which is continuous on \([a,b]\), then, since \( f \) is continuous on \([a,b]\), \( \int_a^b f(x)^2 dx = \int_a^b f(x)g(x) = f(x)^2 dx = 0 \). Problem 34.11, then implies that \( f(x) = 0 \) for all \( x \in [a,b] \).