PERTURBATION THEORY FOR RANDOM WALK IN ASYMMETRIC RANDOM ENVIRONMENT

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ABSTRACT. In this paper the author continues his investigation into the scaling limit of a partial difference equation on the d dimensional integer lattice $\mathbb{Z}^d$, corresponding to a translation invariant random walk perturbed by a random vector field. In a previous paper he obtained a formula for the effective diffusion constant. It is shown here that for the nearest neighbor walk in dimension $d \geq 3$ this effective diffusion constant is finite to all orders of perturbation theory. The proof uses Tutte's decomposition theorem for 2-connected graphs into 3-blocks.

1. Introduction

In this paper we continue the study of the homogenization problem for random walk in asymmetric random environment which was begun in [2]. Let $(\Omega, \mathcal{F}, \mu)$ be a probability space and $b : \Omega \rightarrow \mathbb{R}$ be a bounded measurable function. We assume that $\mathbb{Z}^d$ acts on $\Omega$ by translation operators $\tau_x : \Omega \rightarrow \Omega$, $x \in \mathbb{Z}^d$, which are measure preserving and satisfy the properties $\tau_x \tau_y = \tau_{x+y}$, $x, y \in \mathbb{Z}^d$, $\tau_0 =$ identity. For $i = 1, \ldots, d$, let $e_i \in \mathbb{Z}^d$ be the element with entry 1 in the ith position and 0 in the other positions. Suppose $\gamma \in \mathbb{C}$ and $f : \mathbb{R}^d \rightarrow \mathbb{R}$ is a $C^\infty$ function with compact support. We shall be interested in solutions to the equation on the $\varepsilon$ scaled integer lattice $\mathbb{Z}_\varepsilon^d = \varepsilon \mathbb{Z}^d$ given by

\begin{equation}
(1.1) \quad u_\varepsilon(x, \omega) = \sum_{i=1}^{d} \frac{1}{2d} \left[ u_\varepsilon(x + \varepsilon e_i, \omega) + u_\varepsilon(x - \varepsilon e_i, \omega) \right] - \gamma b(\tau_{x/\varepsilon} \omega) \left[ u_\varepsilon(x + \varepsilon e_1, \omega) - u_\varepsilon(x - \varepsilon e_1, \omega) \right] + \varepsilon^2 u_\varepsilon(x, \omega) = \varepsilon^2 f(x), \quad x \in \mathbb{Z}_\varepsilon^d, \quad \omega \in \Omega.
\end{equation}

In Theorem 1.3 of [2] it was shown that if $\sup_{\omega \in \Omega} |b(\omega)| \leq 1$ and $\gamma < \varepsilon / \sqrt{2d}$ then (1.1) has a unique solution $u_\varepsilon(x, \omega)$ in $L^2(\mathbb{Z}_\varepsilon^d)$ which is also analytic in $\gamma$.

Let $Y_x$, $x \in \mathbb{Z}^d$, be i.i.d. Bernoulli variables, $Y_x = \pm 1$ with equal probability. We now take the function $b(\omega)$ in (1.1) to be defined by $b(\tau_x \omega) = Y_x$, $x \in \mathbb{Z}^d$. In that case Theorem 1.3 of [2] gives an identity for the expectation value of the Fourier transform of the solution $u_\varepsilon(x, \omega)$ of (1.1). For a function $g : \mathbb{Z}_\varepsilon^d \rightarrow \mathbb{C}$ we define its Fourier transform $\hat{g}(\xi)$, $\xi \in [-\pi/\varepsilon, \pi/\varepsilon]^d$, by

\begin{equation}
(1.2) \quad \hat{g}(\xi) = \sum_{x \in \mathbb{Z}_\varepsilon^d} \varepsilon^d g(x) e^{ix \cdot \xi}.
\end{equation}

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Note that if \( \hat{f}_\varepsilon(\xi) \) is the Fourier transform of the function \( f : \mathbf{R} \to \mathbf{R} \) restricted to \( \mathbf{Z}_\varepsilon^d \) then \( \lim_{\varepsilon \to 0} \hat{f}_\varepsilon(\xi) = \hat{f}(\xi), \xi \in \mathbf{R}^d, \) where \( \hat{f}(\xi) \) is the Fourier transform of \( f \). For \( \zeta \in \mathbf{R}^d \) let \( e(\zeta) \) be the \( d \) dimensional vector \( e(\zeta) = (e_1(\zeta), \ldots, e_d(\zeta)) \) with \( e_k(\zeta) = 1 - \varepsilon^{ik} \zeta, k = 1, \ldots, d \). Denote by \( \hat{u}_\varepsilon(\xi) \) the Fourier transform of the expectation of the solution \( u_\varepsilon(x, \omega), \ x \in \mathbf{Z}_\varepsilon^d \), to (1.1). Theorem 1.3 of [2] states that there is a \( d \times d \) matrix \( q_{\gamma, \varepsilon}(\zeta) \zeta \in \mathbf{R}^d \), which is periodic on \([-\pi, \pi]^d\), such that

\[
\hat{u}_\varepsilon(\xi) \left[ 1 + \frac{1}{2d\varepsilon^2} |e(\varepsilon \xi)|^2 - \varepsilon^2 e(\varepsilon \xi)q_{\gamma, \varepsilon}(\varepsilon \xi)e(-\varepsilon \xi) \right] = \hat{f}_\varepsilon(\xi), \xi \in [-\pi/\varepsilon, \pi/\varepsilon]^d.
\]

The matrix \( q_{\gamma, \varepsilon}(\zeta) \) is analytic in \( \gamma \) for \( \gamma < \varepsilon/\sqrt{2d} \) and hence can be written as a convergent power series in this region. By Theorem 1.4 of [2] we may write

\[
q_{\gamma, \varepsilon}(\zeta) = \sum_{m=2}^{\infty} \gamma^{2m} q_{m, \varepsilon}(\zeta).
\]

In [2] it was also shown that in dimension \( d = 1 \) the functions \( q_{m, \varepsilon} \) converge as \( \varepsilon \to 0 \) in the sense that for any compact set \( K \subset \mathbf{R} \) then \( \varepsilon^m q_{m, \varepsilon}(\varepsilon \xi) \) converges uniformly for \( \xi \in K \) to a function \( q_m(\xi) \) as \( \varepsilon \to 0 \). This limit is related to the limits obtained by Sinai[12] and Kesten[7] for one dimensional random walk in random environment.

In this paper we investigate the convergence properties of the matrices \( q_{m, \varepsilon}(\zeta) \) as \( \varepsilon \to 0 \) in dimension \( d > 1 \). In particular we prove the following:

**Theorem 1.1.** For \( d \geq 3 \) there is a constant \( C_d \) depending only on \( d \) such that

\[
|q_{m, \varepsilon}(\zeta)| \leq C_d^m m!, \quad 0 < \varepsilon < 1, \quad \zeta \in \mathbf{R}^d.
\]

Furthermore, the matrix \( q_{m, \varepsilon}(\zeta) \) converges uniformly for \( \zeta \in [-\pi, \pi]^d \) as \( \varepsilon \to 0 \) to a matrix \( q_m(\zeta) \).

In [2] it was shown that the matrix \( q_m(\zeta) = [q_{m, k, k'}(\zeta)], 1 \leq k, k' \leq d \), has only one non-zero entry \( q_{m, 1, 1}(0) \) for \( \zeta = 0 \). Hence taking the formal limit of (1.3) as \( \varepsilon \to 0 \) we obtain the effective homogenized equation for (1.1) in dimension \( d \geq 3 \),

\[
\hat{u}(\xi) \left[ 1 + \frac{\|\xi\|^2}{2d} - (e_1 \cdot \xi)^2 \sum_{m=2}^{\infty} \gamma^{2m} q_{m, 1, 1}(0) \right] = \hat{f}(\xi), \xi \in \mathbf{R}^d.
\]

Theorem 1.1 shows that at the level of formal perturbation theory random walk in asymmetric environment is diffusive at large time for dimension \( d \geq 3 \). It has been rigorously shown [1, 8, 9] that random walk in symmetric environments is diffusive at large time for all dimensions \( d \geq 1 \). Sinai's result [12] shows that the large time behavior of random walk in asymmetric environment is subdiffusive for \( d = 1 \). Fisher [6] and Derrida-Luck [3] predicted diffusive behavior for the large time asymmetric walk if \( d \geq 3 \). There has been some recent rigorous work [1, 10, 13, 14] on (1.1) under the assumption \( < b(\cdot) > \neq 0 \). This situation is very different to the situation studied in Theorem 1.1 since one expects now the drift to dominate diffusion. The methods used in [13, 14] are related to methods used to prove Anderson localisation for the random Schrodinger equation.

The proof of Theorem 1.1 follows a similar strategy to that used to show perturbative renormalization in Euclidean field theories [11]. First one shows by a simple multi-scale decomposition that a large class of Feynman graphs are completely convergent. This is the content of Lemma 2.6 and Corollary 2.1. The basic argument
here goes back to Weinberg [16]. The paper of Feldman et al [4] proves a very general version of Weinberg's theorem. Next one bounds an arbitrary Feynman graph by subdividing the graph into pieces which are completely convergent and then using the cancelation properties of the propagator. This is analogous to the renormalization procedure in Euclidean field theory [5, 11].

The graph subdivision in this paper is implemented by applying Tutte's decomposition theorem for 2-connected graphs into 3-blocks which is described in Chapter IV of [15]. Tutte's theorem does not appear to have been previously used to prove finiteness of Feynman graphs. In this paper we shall use the terminology of [15]. In particular the graphs we consider are multi-graphs with multiple edges but no loops.

2. Proof of Theorem 1.1

For $\eta > 0$, $x \in \mathbb{Z}^d$, let $G_\eta(x)$ be the Green's function which satisfies the equation,

$$G_\eta(x) - \frac{1}{2d} \sum_{i=1}^{d} [G_\eta(x + e_i) + G_\eta(x - e_i)] + \eta G_\eta(x) = \delta(x), \ x \in \mathbb{Z}^d,$$

where $\delta$ is the Kronecker $\delta$ function, $\delta(x) = 0$, $x \neq 0$, $\delta(0) = 1$. If $\hat{G}_\eta(\zeta)$, $\zeta \in [-\pi, \pi]^d$, denotes the Fourier transform of $G_\eta(x)$ as defined by (1.2) with $\varepsilon = 1$ then we have

$$(2.1) \quad \hat{G}_\eta(\zeta) = 1/[\eta + |\zeta|^2/2d], \ \zeta \in [-\pi, \pi]^d.$$

**Lemma 2.1.** Let $h : \mathbb{Z}^d \to \mathbb{C}$ be an exponentially decreasing function with Fourier transform $\hat{h}$. Suppose further there is an $\alpha$ satisfying $0 < \alpha \leq 1$ such that

$$(2.2) \quad |\nabla^k \hat{h}(\zeta)| \leq 1/|\zeta|^{k+1}, \ 0 \leq k \leq d - 1, \ \zeta \in [-\pi, \pi]^d,$$

$$|\nabla^{d-1} \hat{h}(\zeta + \delta) - \nabla^{d-1} \hat{h}(\zeta)| \leq |\delta|/|\zeta|^{d+\alpha}, \ |\delta| < |\zeta|/2, \ \zeta \in [-\pi, \pi]^d.$$

Then there is a constant $C_{d,\alpha}$ depending only on $d, \alpha$ such that

$$(2.3) \quad |h(x)| \leq C_{d,\alpha}/[1 + |x|^{d-1}], \ x \in \mathbb{Z}^d.$$

**Proof.** Since we are assuming $h$ is exponentially decreasing it follows that $\hat{h}$ is $C^\infty$ and $\hat{h}$ is given by the formula

$$(2.4) \quad h(x) = \frac{1}{(2\pi)^d} \int_{[-\pi,\pi]^d} \hat{h}(\zeta) e^{-ix \cdot \zeta} d\zeta, \ x \in \mathbb{Z}^d.$$

From (2.4) and (2.2) with $k = 0$ we see that (2.3) holds if $|x| \leq 1$, whence we shall assume $|x| > 1$. We write

$$(2.5) \quad h(x) = \int_{|x| / |x|} + \int_{|x| / |x|}.$$

It follows again from (2.2) with $k = 0$ that

$$\left| \int_{|x| / |x|} \right| \leq C_d/[1 + |x|^{d-1}],$$
for some constant $C_d'$ depending only on $d$. We estimate the second term on the 
RHS of (2.5) by integrating by parts $d-1$ times. We may assume wlog that 
$x = (x_1, \ldots, x_d)$ satisfies $|x_1| \geq |x|/\sqrt{d}$. We have then that

$$
\int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \hat{h}(\zeta) \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} e^{-ix\zeta} d\zeta \right] d\zeta
\leq \frac{1}{(2\pi)^{d-1}} \int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \left( -i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta)e^{-ix\zeta} d\zeta \right] d\zeta
\leq \frac{1}{(2\pi)^{d-1}} \int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta)e^{-ix\zeta} d\zeta \right] d\zeta
\leq \frac{1}{(2\pi)^{d-1}} \int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta)e^{-ix\zeta} d\zeta \right] d\zeta
$$

where $\mathbf{n} = (n_1, \ldots, n_d)$ is the inward pointing unit normal on the sphere $|\zeta| = 1/|x|$. From (2.2) with $k = 0, \ldots, d-2$ it follows that the surface integral on $\{|\zeta| = 1/|x|\}$ in the last expression is bounded by the RHS of (2.3). We are left then to estimate the 
volume integral on $\{|\zeta| > 1/|x|\}$. From (2.2) with $k = d-1$ we obtain a bound which 
logarithmically larger than the RHS of (2.3). We use the second inequality of (2.2) to 
improve this. For $\delta \in \mathbb{R}^d$ let $S_\delta = \{\zeta \in \mathbb{R}^d : \zeta + \delta \in [-\pi, \pi]^d, |\zeta + \delta| > |x|^{-1}\}$. Evidently we have that

$$
\int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta)e^{-ix\zeta} d\zeta \right] d\zeta
\leq \int_{S_\delta} \left[ \int_{S_\delta} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta + \delta)e^{-ix(\zeta + \delta)} d\zeta \right] d\zeta,
$$

for any $\delta \in \mathbb{R}^d$. We take $\delta = (\delta_1, \ldots, \delta_d)$ with $\delta_1 = \pi/x_1, \delta_j = 0, j \neq 1$. Then the 
LHS of (2.6) is given by

$$
\int_{|\zeta| > 1/|x|} \left[ \int_{|\zeta| > 1/|x|} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta)e^{-ix\zeta} d\zeta \right] d\zeta
\leq \int_{S_\delta} \left[ \int_{S_\delta} \left( i \frac{\partial}{\partial \zeta_1} \right)^{d-1} \hat{h}(\zeta + \delta)e^{-ix(\zeta + \delta)} d\zeta \right] d\zeta
$$

From (2.2) with $k = d-1$ the last two integrals in (2.7) are bounded by a constant 
$C_d$ depending only on $d$. From the second inequality of (2.2) the first integral in 
(2.7) is bounded by a constant $C_{d,\alpha}$ depending only on $d, \alpha$.

**Lemma 2.2.** For $\eta > 0$ let $K_\eta(x)$ be the function 
$K_\eta(x) = G_\eta(x - e_1) - G_\eta(x + e_1)$, $x \in \mathbb{Z}^d$. Suppose $g_k : \mathbb{Z}^d \to \mathbb{C}$, $k = 1, \ldots, n$ are exponentially decreasing 
functions which satisfy $g_k(x) = -g_k(-x)$, $|g_k(x)| \leq 1/[1 + |x|^{d-1}]^3$, $x \in \mathbb{Z}^d$, $1 \leq k \leq n$. Let $h : \mathbb{Z}^d \to \mathbb{C}$ be the convolution,

$$
h = K_\eta \ast g_1 \ast K_\eta \ast g_2 \ast \cdots \ast K_\eta \ast g_n \ast K_\eta.
$$

Then $h$ is exponentially decreasing and satisfies $h(x) = -h(-x)$, $|h(x)| \leq C_d^\eta/[1 + |x|^{d-1}]$, $x \in \mathbb{Z}^d$, for some constant $C_d$ depending only on $d$. 

Proof. It is evident that $h$ is exponentially decreasing and that $h(x) = -h(-x)$. To obtain the bound on $|h(x)|$ we use Lemma 2.1. We have that

$$
\hat{h}(\zeta) = \hat{K}_n(\zeta) \prod_{k=1}^n \hat{g}_k(\zeta).
$$

From (2.1) we have that

$$
\hat{K}_n(\zeta) = 2i \sin \zeta / [\eta + |e(\zeta)|^2 / 2d].
$$

Since $g_k$ is an odd function we have

$$
\hat{g}_k(\zeta) = i \sum_{x \in Z^d} g_k(x) \sin (x \cdot \zeta).
$$

Hence if $d > 3$ there is a constant $C_d$ depending only on $d$ such that

$$
|\nabla^k \hat{h}(\zeta)| \leq C_d^{a+1} |\zeta|^{k+1}, \quad 0 \leq k \leq d.
$$

The bound on $|h(x)|$ follows now from Lemma 2.1 if $d > 3$. For $d = 3$ the first inequality of (2.2) holds. The second inequality of (2.2) also holds for $\alpha < 1$ by observing that

$$
\left| \sum_{x \in Z^d} |x|^2 g_k(x) [\sin (x \cdot \zeta + x \cdot \delta) - \sin x \cdot \zeta] \right| \leq C_\alpha |\delta|^{a},
$$

where $C_\alpha$ depends only on $\alpha$. $\square$

By Theorem 1.4 of [2] the $d \times d$ matrix $q_{m,e}(\zeta) = [q_{m,e,k,k'}(\zeta)]$ defined by (1.4) has nonzero entries only for $k^2 = 1$, $1 \leq k \leq d$. The function $q_{m,e,1,k}(\zeta)$ is a sum of terms $K_{e,k}(G,e,\zeta)$ where $G$ is a connected graph with $2m$ edges, at most $m$ vertices, and $e$ is a distinguished edge of $G$. Denoting by $V[G]$ the set of vertices of $G$ and $E[G]$ the set of edges, then $K_{e,k}(G,e,\zeta)$ is given by the formula,

$$
K_{e,k}(G,e,\zeta) = \sum_{y \in Z^d, z \in V[G]} (y \cdot e) \delta(y) \prod_{e' \in E[G], e' \neq e} K_e \left( y - y' \right)
$$

$$
\int_0^1 \exp[-ity \cdot \zeta] \, dt / \int_0^1 \exp[-iye \cdot \zeta] \, dt.
$$

In (2.8) the notation $f_+$ and $f_-$ are used for the vertices of an edge $f \in E[G]$. The formula (2.8) generalizes Lemma 5.4 of [2]. The graphs $G$ which occur have the property that the degree of every vertex is divisible by 4. The following proposition is therefore the main step in proving Theorem 1.1.

**Proposition 2.1.** Let $G$ be a connected graph such that the degree of every vertex is divisible by 4. Then there is a constant, $C_d$ depending only on $d$, such that

$$
|K_{e,k}(G,e,\zeta)| \leq C_d |E[G]|, \quad 0 < \varepsilon \leq 1, \quad \zeta \in C_1.
$$

We shall prove Proposition 2.1 in a series of lemmas. Our first goal will be to show that we may assume wlog that $G$ is 2-connected. To do this we need an elementary result from graph theory.

**Lemma 2.3.** Let $G$ be a connected graph and suppose that the degree of all but one vertex of $G$ is divisible by 4. Then all vertices of $G$ have even degree and $G$ has an odd number of edges.
Proof. For \( v \in V[G] \) let \( d(v) \) be the degree of the vertex \( v \). Then there is the identity
\[
\sum_{v \in V[G]} d(v) = 2|E[G]|.
\] (2.11)

The result easily follows. \( \Box \)

**Lemma 2.4.** Let \( G \) be a connected graph such that the degree of every vertex is divisible by 4. Let \( B \) be a block of the decomposition of \( G \) into 2-connected components. If \( K_e(G,e) \neq 0 \) then the degree of every vertex of \( B \) is divisible by 4.

Proof. We shall use induction on the number of blocks in the block decomposition of \( G \). We may assume wlog that \( G \) has at least 2 blocks in which case there is an endblock \( B_0 \) which does not contain the distinguished edge \( e \in E[G] \). Let \( v_0 \in V[B_0] \) be the cut vertex of \( B_0 \) in \( G \) and \( K_e(B_0) \) be given by
\[
K_e(B_0) = \sum_{\{y_e \in \mathbb{Z}^d : v \in V[B_0]\}} \delta(y_{v_0}) \prod_{e \in E[B_0]} K_e(x_{y_{e+}} - y_{e-}).
\]

Then it is clear that \( K_e(G,e) = K_e(G \setminus B_0, e)K_e(B_0) \). Since \( K_e(G,e) \neq 0 \) it follows that \( K_e(B_0) \neq 0 \) and \( K_e(G \setminus B_0, e) \neq 0 \). If \( B_0 \) has an odd number of edges then \( K_e(B_0) = 0 \) since \( K_{e_0}(x) = -K_{e_0}(-x) \), \( x \in \mathbb{Z}^d \). Hence \( B_0 \) has an even number of edges and it also has the property that the degree of every vertex other than \( v_0 \) is divisible by 4. It follows then from Lemma 2.1 that the degree of \( v_0 \in V[B_0] \) is divisible by 4 whence the degree of \( v_0 \in V[G \setminus B_0] \) is also divisible by 4. \( \Box \)

Next we show that if \( G \) is 3-edge connected then the result of Proposition 2.1 holds.

**Lemma 2.5.** Suppose the graph \( G \) is 3-edge connected, the degree of each vertex of \( G \) is even and at least 4. Let \( G' \) be the graph obtained from \( G \) by the contraction of 2 vertices. Then \( G' \) is also 3-edge connected and the degree of each vertex of \( G' \) is even and at least 4.

Proof. Obvious. \( \Box \)

**Lemma 2.6.** Suppose the graph \( G \) is 3-edge connected, the degree of each vertex of \( G \) is even and at least 4. Let \( e \in E[G] \) and for each edge \( e' \in E[G] \setminus \{e\} \) let \( n_{e'} \) be an arbitrary non-negative integer. Then there is a constant \( C_d \) depending only on \( d \) such that there is the inequality
\[
\sum_{\{y_e \in \mathbb{Z}^d : v \in V[G]\}} |y_{e-}| \delta(y_{e+}) \prod_{e \in E[G], e' \neq e} 2^{-n_{e'}} d^{1/2} \exp \left[-2^{-n_{e'}} |y_{e'+} - y_{e'-}| \right] \leq C_d^{[V[G]]}.
\] (2.12)

Proof. The set \( \{n_{e'} : e' \in E[G] \setminus \{e\}\} = \{N_1, N_2, ..., N_k\} \) with \( 0 \leq N_1 < N_2 < ... < N_k \). Let \( G_1 = G \) and consider the graph \( G'_1 \) with edges \( e' \in E[G] \setminus \{e\} \) satisfying \( n_{e'} = N_1 \). Contract each of the connected components of \( G'_1 \) to a single vertex. Thus we obtain a graph \( G_2 \) from \( G_1 \) by the contraction process. By Lemma 2.5 the graph \( G_2 \) is 3-edge connected, the degree of each vertex of \( G_2 \) is even and at least 4. Next we consider the subgraph \( G'_2 \) of \( G_2 \) corresponding to the edges \( e' \in E[G] \setminus \{e\} \) satisfying \( n_{e'} = N_2 \). We contract each of the connected components of \( G'_2 \) to a single vertex, whence we obtain a graph \( G_3 \) from \( G_2 \). Proceeding in this manner we construct graphs \( G_1, ..., G_k \) and graphs \( G'_1, ..., G'_k \) where \( G'_j \) is a subgraph of
$G_j$, $1 \leq j \leq k$. Note that each edge of $G'_k$ corresponds to an edge $e' \in E[G]\{e\}$ with $n_{e'} = N_k$. Since $G'_k$ is 3-edge connected and $G'_k$ differs from $G_k$ by at most one edge corresponding to $e$ it follows that $G'_k$ is connected.

Next we construct a spanning tree $T$ for the graph $G$. First let $T_k$ be a spanning tree for $G'_k$. Now for each component of $G'_{k-1}$ we can form a spanning tree. The graph $G_{k-1}$ is obtained from $G_k$ by splitting certain vertices into the components of $G'_{k-1}$. Thus the spanning trees for each component of $G'_{k-1}$ together with $T_k$ yield a spanning tree $T_{k-1}$ for $G_{k-1}$. Similarly we obtain spanning trees $T_j$ for $G_j$, $1 \leq j \leq k$. We put $T = T_1$.

For a subgraph $H$ of $G$ and $j = 1, \ldots, k$ let $E_H[N_j] = \{e' \in E[H]\{e\} : n_{e'} = N_j\}$. We define an integer $k_0$, $1 \leq k_0 \leq k$ as follows: Let $e_-$ and $e_+$ be the vertices of the distinguished edge $e \in E[G]$. If $e_-$ and $e_+$ correspond to different vertices of $G_k$ in the contraction process then $k_0 = k$. If they correspond to the same vertex of $G_k$ there is a $k_0 < k$ such that they correspond to the same vertex of $G_j$ for $j > k_0$ but to different vertices of $G_{k_0}$. This defines $k_0$ uniquely.

We obtain inequalities relating the number of edges in $E_T[N_j]$ to the number of edges in $E_G[N_j]$. It is easy to see in fact that

$$
2 \sum_{j=r}^k |E_T[N_j]| + 2 \leq \sum_{j=r}^k |E_G[N_j]|, \ r > k_0,
$$

$$
2 \sum_{j=r}^k |E_T[N_j]| + 1 \leq \sum_{j=r}^k |E_G[N_j]|, \ r \leq k_0.
$$

Now there exists a path $P$ in $T$ from $e_-$ to $e_+$ such that every edge of the path is in $E_G[N_j]$ for some $j \leq k_0$. It follows then from (2.13) that the LHS of (2.12) is bounded by

$$
\sum_{f \in I[P]} \sum_{\{y_0, e \in Z^d, e \in V[T]}} \delta(y_{e_+})|y_{f_+} - y_{f_-}| 2^{-3n_f} d^{d/2} \exp \left[ -2^{-n_f} |y_{f_+} - y_{f_-}| \right] \prod_{f' \in I[T]\{f\}} 2^{-n_{f'}} \exp \left[ -2^{-n_{f'}} |y_{f'_+} - y_{f'_-}| \right].
$$

It is easy to see that the last expression is bounded by $C_{d}^{I[V[T]]}$ for some constant $C_{d}$.

**Corollary 2.1.** Suppose the graph $G$ is 3-edge connected, the degree of each vertex of $G$ is even and at least 4. Let $e \in E[G]$ and for each edge $e' \in E[G]\{e\}$ let $K_{e'} : Z^d \to R$ be a function which satisfies the inequality,

$$
|K_{e'}(x)| \leq 1/\left[ 1 + |x|^{d-1} \right], \ x \in Z^d.
$$

Then there is a constant $C_d$ depending only on $d$ such that

$$
\sum_{\{y_0, e \in Z^d, e \in V(G)\}} |y_{e_+}| \delta(y_{e_+}) \prod_{e' \in E[G]\{e\}} |K_{e'}(y_{e'_+} - y_{e'_-})| \leq C_d^{I[E[G]]}.
$$

**Proof.** From (2.14) one sees there is a constant $C_d$ depending only on $d$ such that

$$
|K_{e'}(x)| \leq C_d \sum_{n=0}^{\infty} 2^{-n(d-1)} \exp \left[ -2^{-n} |x| \right], \ x \in Z^d.
$$

The result follows from Lemma 2.6 since $d - 1 > d/2$ for $d > 2$. 

Next we prove a more general result than Corollary 2.1.

**Lemma 2.7.** Let $G$ be a graph as in Corollary 2.1 with associated functions $K_{e'}$, $e' \in E[G] \setminus \{e\}$ which satisfy (2.14). Then there is a constant $C_d$ depending only on $d$ such that

$$
\sum_{\{y_-, y_+ \in \mathbb{Z}^d, v \in V[G]\}} \delta(y_--x) \delta(y_+) \prod_{e' \in E[G] \setminus \{e\}} |K_{e'}(y_{e'_+} - y_{e'_-})| \leq C_d^{E[G]} / [1 + |x|^{d-1}]^3, \ x \in \mathbb{Z}^d.
$$

**Proof.** The result follows from a generalization of Lemma 2.6 in the same way as Corollary 2.1. Using the notation of Lemma 2.6 we need to show that for some constant $C_d$ depending only on $d$, there is the inequality,

$$
\sum_{\{y_-, y_+ \in \mathbb{Z}^d, v \in V[G]\}} \delta(y_--x) \delta(y_+) \prod_{e' \in E[G] \setminus \{e\}} 2^{-n_{e'}} d/2 \exp \left[-2^{-n_{e'}} |y_{e'_+} - y_{e'_-}| \right] \leq C_d^{E[G]} 2^{-3N_{k_0}} d/2 \exp \left[-2^{-|N_{k_0+1}|} |x| \right], \ x \in \mathbb{Z}^d.
$$

The inequality (2.16) is obtained by summing out all variables except those attached to vertices along the path $P$ in $T$ from $e_-$ to $e_+$. Now for $e' \in E[P]$ we have $n_{e'} \leq N_{k_0}$ and at least one $e'$ has $n_{e'} = N_{k_0}$. Now (2.16) follows by first taking out the exponential factor $\exp[-2^{-|N_{k_0+1}|} |x|]$ using the fact that

$$
\exp \left[-\sum_{e' \in E[P]} 2^{-|n_{e'}+1|} |y_{e'_+} - y_{e'_-}| \right] \leq \exp[-2^{-|N_{k_0+1}|} |x|].
$$

Then we remove an edge $e'$ satisfying $n_{e'} = N_{k_0}$ and sum over the remaining variables. To finish the proof of (2.15) we simply observe that from (2.16) the LHS of (2.15) is bounded above by

$$
C_d^{E[G]} \sum_{\{e_1, e_2, e_3 \in E[G] \setminus \{e\}\}} \left( \prod_{e' \in E[G] \setminus \{e\}} \sum_{n_{e'}=0}^{\infty} \right) \prod_{e' \in E[G] \setminus \{e, e_1, e_2, e_3\}} 2^{-n_{e'}(d-1) - d/2} \prod_{i=1}^{3} 2^{-n_{e_i}(d-1)} \exp \left[-2^{-|n_{e_i}+3|} |x| \right],
$$

where $C_d$ depends only on $d$ and the sum over $e_1, e_2, e_3$ is over all distinct sets of edges $\{e_1, e_2, e_3\} \subset E[G] \setminus \{e\}$. \hfill $\square$

We turn to the proof of Proposition 2.1 in the general case where by Lemma 2.4 we may assume that $G$ is 2-connected. We also assume that $G$ is not 3-connected since in that case Proposition 2.1 already follows from Corollary 2.1. Hence $G$ has a non trivial Tutte decomposition [15] into 3-blocks. The 3-blocks of $G$ are either 3-connected graphs, $n$-linkages with $n \geq 3$ or cycles. The following lemma shows that if there are no 3-blocks which are cycles then $G$ is 3-edge connected whence Proposition 2.1 follows from Corollary 2.1.

**Lemma 2.8.** Let $G$ be a graph, $T$ a tree graph and $W_t$, $t \in V[T]$, be a system of subgraphs of $G$ with the properties:

(a) $\bigcup_{t \in V[T]} W_t = G$,

(b) every edge of $G$ belongs to exactly one $W_t$. 


(c) If \( t, t' \in T \) are adjacent then \( W_t \cap W_{t'} \) consists of precisely 2 vertices. If \( t, t'' \in T \) are not adjacent then \( W_t \cap W_{t''} \subset W_t \cap W_{t'} \) where \( t' \) is the vertex adjacent to \( t \) which lies on the path in \( T \) from \( t'' \) to \( t \).

For \( t \in T \) let \( W_t^* \) be the graph obtained from \( W_t \) by adding an edge joining the pair of vertices in \( W_t \cap W_{t'} \) for all \( t' \in T \) adjacent to \( t \). Suppose that the graphs \( W_t^* \) are 3-edge connected for all \( t \in T \). Then \( G \) is 3-edge connected.

Proof. It is easy to see that \( G \) is 3-edge connected if \( T \) has just 2 vertices. More generally the result follows by induction on contraction of an edge of \( T \). \( \square \)

Proof of Proposition 2.1: If \( G \) has no 3-blocks which are cycles then the result follows from Lemma 2.6 and Corollary 2.1. Suppose then \( G \) has exactly one 3-block which is a cycle. Let \( T \) be the tree graph corresponding to the Tutte decomposition which has 3-blocks of \( G \) as its vertices. Let \( t_0 \in V[T] \) be the vertex corresponding to the 3-block which is a cycle. Since the degree of every vertex of \( G \) is at least 4, the vertex \( t_0 \) cannot be an end vertex of \( T \) whence \( T \setminus \{t_0\} \) has at least 2 components. Suppose \( T \setminus \{t_0\} \) has \( m \) components \( T_1, \ldots, T_m \). These correspond to \( m \) virtual edges \( f_1, \ldots, f_m \) of the cycle, which we assume are in order on the cycle with \( f_{j+1} \) following \( f_j \), \( 1 \leq j \leq m - 1 \). Note that the \( f_j \), \( 1 \leq j \leq m \), do not necessarily constitute all edges of the cycle since there can be edges which belong to \( G \). Using the notation of Lemma 2.8 we put \( H_j = \cup_{t \in T_j} W_t, 1 \leq j \leq m \). Let \( H_j \) be the graph consisting of the union of \( H_j \) and the edge \( f_j \), \( 1 \leq j \leq m \). By Lemma 2.8 \( H_j \) is 3-edge connected. Suppose now that the cycle has \( m \) edges whence \( G = \cup_{j=1}^m H_j \). Since each \( H_j \) is 3-edge connected it is clear that \( G \) is 3-edge connected whence the result follows from Corollary 2.1. Alternatively let us assume that the cycle has more than \( m \) edges. Then the graph with edges \( f_1, \ldots, f_m \) has \( k \leq m \) components \( F_1, \ldots, F_k \) and each component \( F_i \) is a simple path with 2 vertices of degree 1 which we denote by \( F_i^+ \) and \( F_i^- \), \( 1 \leq i \leq k \). We put \( G_i = \cup_{f_i \in F_i} H_{f_i^+}, 1 \leq i \leq k \), and let \( G_i \) be the graph which is the union of \( G_i \) and the edge \( F_i^+ F_i^- \), \( 1 \leq i \leq k \). Since each \( H_j \) is 3-edge connected it follows that \( G_i \) is 3-edge connected, \( 1 \leq i \leq k \). One also easily sees that the degree of every vertex of \( G_i \) is divisible by 4.

We consider how to write \( K_{e,k}(G,e,\zeta) \) in terms of a convolution of functions defined by the graphs \( G_i \), \( 1 \leq i \leq k \). For \( 1 < i \leq k \) we define functions \( g_i(x), x \in \mathbb{Z}^d \), by

\[
g_i(x) = \sum_{y \in \mathbb{Z}^d, \exists v \in V[G_i]} \delta(y_{F_i^-} - x) \delta(y_{F_i^+}) \prod_{e' \in E[G_i]} K_{e'}(y_{e' +} - y_{e'}) .
\]

It is evident that \( g_i(-x) = -g_i(x) \). By Lemma 2.7 one also has that

\[
|g_i(x)| \leq C_d^{E[G_i]} / [1 + |x|^{d-1}]^3, \quad x \in \mathbb{Z}^d .
\]

First we consider the case when \( e \) is an edge of the cycle. We may assume \( e_+ = F_k^+ \), \( e_- = F_k^- \). Let \( h \) be the function,

\[
h(x) = g_1 * K_{e_2} * g_2 * \cdots * K_{e_k} * g_k(x), \quad x \in \mathbb{Z}^d .
\]

Then there is the identity,

\[
\int_0^1 \exp[-ite_k \cdot \zeta] \ dt \ K_{e,k}(G,e,\zeta) = \int_0^1 i \frac{\partial^n}{\partial \zeta_k}(t \zeta) \ dt .
\]
By the argument of Lemma 2.2 there is a constant $C_d$ depending only on $d$ such that

$$|\nabla h(\zeta)| \leq C_d^{[G]}|\zeta|, \quad \zeta \in \mathcal{C}.$$ 

Hence the inequality (2.9) of Proposition 2.1 holds. Next we consider the case when $e$ is not an edge of the cycle so we may wlog assume that $e \in E[G]_1$. Now for $e' \in E[G]_1$, $e' \neq e$, $F_{i}^{-}F_{i}^{+}$ we have $K_{e'}(x) = K_{e}(x)$, $x \in \mathbb{Z}^d$. We define now $K_{f}(x)$ for $f = F_{i}^{-}F_{i}^{+}$ by

$$K_{f}(x) = K_{e_{1}} * g_{2} * K_{e_{2}} * g_{3} * \cdots * K_{e_{k}} * g_{k} * K_{e_{k}}(x), \quad x \in \mathbb{Z}^d. \tag{2.17}$$

By Lemma 2.2 we have that $K_{f}(x) = -K_{f}(-x)$, $x \in \mathbb{Z}^d$ and

$$|K_{f}(x)| \leq C_d^{[E[G]]}[|E[G]|] / \left[ 1 + |x|^{d-1} \right], \quad x \in \mathbb{Z}^d,$$

for some constant $C_d$ depending only on $d$. Hence by Corollary 2.1 applied to the graph $G_1$ with distinguished edge $e$ the inequality (2.9) of Proposition 2.1 follows.

Next we consider the situation where the Tutte decomposition of $G$ has more than one 3-block which is a cycle. Letting $T$ be the tree graph corresponding to the Tutte decomposition of $G$, we can easily see that there exists a vertex $t_0 \in T$ which has the properties:

(a) $t_0$ corresponds to a 3-block which is a cycle.
(b) Exactly one component of $T \backslash \{t_0\}$ contains all 3-blocks which are cycles.
(c) The edge $e$ belongs to a 3-block which is contained in the unique component defined in (b).

Now we proceed exactly as before to eliminate $t_0$ and thus reducing the number of 3 blocks which are cycles, whence (2.9) follows by induction.

We finally prove that the limit (2.10) exists. Since for $d \geq 2$ $\lim_{\varepsilon \to 0} K_{e_{1}}(x) = K_{0}(x)$, $x \in \mathbb{Z}^d$, (2.10) follows from Corollary 2.1 in the case $G$ is 3-edge connected. Consider the situation when $G$ has exactly one 3-block which is a cycle. We denote the kernel $K_{f}$ of (2.17) by $K_{f,\varepsilon}$ to denote the dependence on $\varepsilon$. It is clear from the proof of Lemma 2.2 that $\lim_{\varepsilon \to 0} K_{f,\varepsilon}(x) = K_{f,0}(x)$. Hence (2.10) follows as before. A similar argument gives (2.10) when $G$ has more than one 3-block which is a cycle.

**Proof of Theorem 1.1:** From Lemma 5.3 and 5.4 of [2] it follows that there is a universal constant $C > 1$ such that $q_{m, \varepsilon}(\zeta)$ is a sum of less than $C^m m!$ terms each of which has the form (2.8) and such that the number of edges of the graph is $2m$. The result follows now from Proposition 2.1.

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