Solutions to Requested Problems
From Ross, Chaps. 4, 5 and 7.

These are problems which came up either by email or in the problem sessions which I had to think a bit about before solving. I promised to put them up on the web page. These are, in fact, three of the problems already posted on the web as "Optional further examples of problems from earlier chapters of the text, as per student requests (for preparation for the final examination)". Sounds like it is a good time to look over them!

Chap. 4, prob. 24. Game Theory: The Minimax Theorem.

We are given two players, $A$ and $B$, and $A$ picks one of two numbers, 1 or 2. $B$ then has to guess what number $A$ picked. If $B$ guesses number $i$ correctly ($i$ can be 1 or 2), then $B$ gets $i$ dollars from $A$. If $B$ is wrong, $A$ gets $3i$ dollars from $B$. One strategy is for $B$ to randomize his guess, saying 1 with probability $p$ and 2 with probability $1-p$. The problem will be about what strategy the two players should follow. We have already said $B$ will try to randomize. If $A$ were to try to avoid losing too much to $B$, you might think he should always pick 1, but then that gives $B$ something to predict reliably, and he can count on winning 1 dollar from $A$ every turn. So you see there is a problem of picking a strategy without your opponent being able to read exactly what it is and therefore being able to anticipate your move. That is also the reason why a random strategy might be useful.

Set $G$ equal to the random variable "$B$’s gain. So, with $B$ using this random strategy, we get that

$$E(G|A \text{ chooses } 1) = p \cdot 1 - \frac{3}{4}(1 - p),$$

and

$$E(G|A \text{ chooses } 2) = (1 - p) \cdot 2 - \frac{3}{4}p.$$ 

First notice that

$$E(G|A \text{ chooses } 1) = E(G|A \text{ chooses } 2)$$
at $p = \frac{11}{18}$. Then

$$\min\{E(G|A\text{ chooses 1}), E(G|A\text{ chooses 2})\} = \begin{cases} p \cdot 1 - \frac{3}{4} \cdot (1 - p), & p \leq \frac{11}{18}, \\ (1 - p) \cdot 2 - \frac{3}{4} \cdot p, & p \geq \frac{11}{18}. \end{cases}$$

The maximum of this is attained at $p = \frac{11}{18}$ and is equal to $\frac{23}{72}$. (You can see this by inspection from the graphs.) The “maximin” is $\frac{23}{72}$.

Similarly, you can see that if we take the point of view of $A$ trying to minimize her losses, we would look at $G$ as her loss, and calculate first that

$$\max\{E(G|B\text{ chooses 1}), E(G|B\text{ chooses 2})\} = \begin{cases} q \cdot 1 - \frac{3}{4} \cdot (1 - q), & q \geq \frac{11}{18}, \\ (1 - q) \cdot 2 - \frac{3}{4} \cdot q, & q \leq \frac{11}{18}. \end{cases}$$

Again, by inspection, the minimum of this is attained when $q = \frac{11}{18}$, and the minimum is $\frac{23}{72}$, which is the “minimax”, and is equal, as predicted by Von Neumann, to the maximin.

**Chap. 5, prob. 29. Asymmetric Multiplicative Random Walk (Biased Stock Market)**

We are given a model of stock price randomness: if it sells at $s$ today, with probability $p$ it will sell at $us$ tomorrow, and with probability $1 - p$ it will sell at $ds$. So, $u = 1 + r_+$, where $r_+$ is the fractional gain in the day, and $d = 1 - r_-$, where $r_-$ is the fractional loss on the day. The model is asymmetric in that $r_+$ is not assumed equal to $r_-$. Let $X$ be the random variable which is $\log u$, if the stock goes up, and $\log d$ if the stock goes down. Let $X_i$ be i.i.d.’s, all $\sim X$. It is easy to check that the price of the stock after $n$ trading days of the sort described is just $P_n = e^{S_n} \cdot s$, where, as usual, $S_n = X_1 + \ldots + X_n$. The question asks for $\Prob(\frac{P_{1000}}{s} \geq 1.3)$, at least approximately. But, taking log’s, this is the same as asking $\Prob(S_{1000} \geq \log 1.3)$. Now to estimate this last, we can use DeMoivre’s theorem (or the CLT theorem now), where we have to subtract from $S_{1000}$ its mean, which is $1000 \cdot \{(0.52) \cdot \log 1.012 + (0.48) \cdot \log 0.990\} = 1.3758$ and divide by its standard deviation, which would be $\sqrt{1000 \cdot V(X_1)}$, since they are iid’s. This is just $\sqrt{1000 \cdot V(X_1)} = 0.35$ (calculator). Thus, we want to calculate $\Prob\left(\frac{S_{1000} - 1.3758}{0.35} \geq \frac{\log 1.3 - 1.3758}{0.35}\right) \approx 2$.
\[ \Pr(Z \geq 0.2624 - 1.3758 \cdot 0.35) = \Pr(Z \geq -3.18) = \Pr(Z \leq 3.18) = \Phi(3.18) = 0.993, \] from Table 5.1 in the book, page 203.

**Chap. 7, prob. 24. Splitting Pills.**

The basic story is that there are \( n + m \) pills in a bottle, \( n \) of which are small, and \( m \) of which are large. Each day the patient draws a pill out at random, and if it is a small pill, takes it, and if it is a large pill, breaks it in two, takes one half and replaces the other half in the bottle where it is then considered a small pill.

**a)** Let \( X \) be the random variable, “the number of small pills in the bottle when one has drawn the last big pill and put the uneaten half back into the bottle.” What is \( E(X) \), the expected number of small pills left in the bottle after the last big one has been broken?

**Solution of a)** We can follow the hint, and look at an indicator variables method as in example 7.2.m in the textbook. If you follow this whole experiment out to the end, one will have \( 2m + n \) days worth of pills, and they can be categorized into three kinds: the original small ones (\( n \) of these), the first halves of all the big pills which were broken in two (\( m \) of these), and the \( m \) big pills (which are broken in two and only half eaten). Let \( S_j, j = 1, \ldots, n, \) be indicator random variables where \( S_j = 1 \) if the \( j \)-th small pill gets drawn after all the big pills. Let \( F_i, i = 1, \ldots, m, \) be indicator random variables where \( F_i = 1 \) if the small pill derived from breaking the \( i \)-th big pill is drawn after all the big pills. Then we have a counting formula:

\[
X = S_1 + \ldots + S_n + F_1 + \ldots + F_m,
\]

and taking expectations, we get

\[
E(X) = E(S_1) + \ldots + E(S_n) + E(F_1) + \ldots + E(F_m) = nE(S_1) + mE(F_1).
\]

Arguing as in the textbook, if we consider the \( m \) big pills \( b_1, \ldots, b_m \) and the first small pill \( s_1 \), the probability that \( s_1 \) will be drawn after all of \( b_1, \ldots, b_m \) is \( \frac{1}{m+1} \), since \( s_1 \) is equally likely to be drawn at any of the \( m+1 \) possible positions in the ordering of these \( m + 1 \) pills. This gives us \( E(S_1) = \Pr(b_1 = 1) = \frac{1}{m+1} \). The case of \( E(F_1) \) is a little trickier. Here one considers the \( m + 1 \) pills \( b_1, \ldots, b_m, f_1 \), but there is a constraint in the order here, since by the design of the experiment, \( b_1 \) must always be drawn before \( f_1 \) (\( f_1 \) doesn’t exist yet until \( b_1 \) is drawn and broken in two). Therefore we only have to consider the
order of the \( m \) pills \( b_2, \ldots, b_m, f_1 \), and as earlier, the probability that \( f_1 \) will be drawn last is just \( \frac{1}{m} \). Putting all this together, we get

\[
E(X) = nE(S_1) + mE(F_1) = n \frac{1}{m+1} + m \frac{1}{m} = \frac{n}{m+1} + 1. 
\]

b) Now let \( Y \) be the random variable, “the day on which the last big pill was drawn”, where we start at 1 the day we draw the first pill, etc. What is \( E(Y) \), that is, when do we expect to draw the last big pill?

**Solution of b)** This one is easy, because it follows from what has already been done: there are \( 2m + n \) days worth of pills in the bottle originally, and if there are \( X \) small pills left in the bottle when one draws the last big pill, there will be exactly that many days until one finishes all the pills. Therefore, one had to have picked the last big pill on day \( Y = 2m + n - X \). Taking expectations gives

\[
E(Y) = 2m + n - E(X) = 2m + n - \frac{n}{m+1} - 1. 
\]