Problem 1. (5 pts.) Let \( \mathbf{u} \) and \( \mathbf{v} \) be nonzero vectors in the plane. Find \( \mathbf{u} \cdot (\mathbf{v} - \text{proj}_\mathbf{u} \mathbf{v}) \).

**Solution.** Recall that the length of the projection of \( \mathbf{v} \) onto \( \mathbf{u} \) is \( |\mathbf{v}| \cos(\theta) \), where \( \theta \) is the angle between \( \mathbf{u} \) and \( \mathbf{v} \). Thus the length of the projection is \( \mathbf{u} \cdot \mathbf{v} \frac{1}{|\mathbf{u}|^2} \), and by multiplying this against a unit vector in the \( \mathbf{v} \) direction we get the projection: \( \text{proj}_\mathbf{u} \mathbf{v} = \mathbf{u} \cdot \mathbf{v} \frac{1}{|\mathbf{u}|^2} \mathbf{u} \). Thus,

\[
\mathbf{u} \cdot (\mathbf{v} - \text{proj}_\mathbf{u} \mathbf{v}) = \mathbf{u} \cdot (\mathbf{v} - \mathbf{u} \cdot \mathbf{v} \frac{1}{|\mathbf{u}|^2} \mathbf{u})
\]
\[
= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \frac{1}{|\mathbf{u}|^2} \mathbf{u} \cdot \mathbf{u}
\]
\[
= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v} \frac{|\mathbf{u}|^2}{|\mathbf{u}|^2}
\]
\[
= \mathbf{u} \cdot \mathbf{v} - \mathbf{u} \cdot \mathbf{v}
\]
\[
= 0.
\]

Is this surprising? We can also deduce this intuitively: \( \text{proj}_\mathbf{u} \mathbf{v} \) is the part of \( \mathbf{v} \) that lies in the direction of \( \mathbf{u} \). Therefore, \( \mathbf{v} - \text{proj}_\mathbf{u} \mathbf{v} \) is the part of \( \mathbf{v} \) that is perpendicular to \( \mathbf{u} \). So \( \mathbf{u} \cdot (\mathbf{v} - \text{proj}_\mathbf{u} \mathbf{v}) \) is the dot product of \( \mathbf{u} \) with a vector perpendicular to it, which must be zero. ♦

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Problem 2. (5 pts.) Find the equation of the plane passing through three points \( P(2,1,1), Q(3,-1,1), \) and \( R(4,1,-1) \).

**Solution.** To write the equation of a plane, we need a normal vector and a point. We have lots of points, so we just need to find a normal vector. Because \( P, Q, \) and \( R \) are in the plane, a vector perpendicular to both \( \overrightarrow{PQ} \) and \( \overrightarrow{PR} \) will be just the ticket. We can find this with the cross-product. \( \overrightarrow{PQ} = <1,-2,0> \) and \( \overrightarrow{PR} = <2,0,-2> \), so

\[
\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & -2 & 0 \\ 2 & 0 & -2 \end{vmatrix} = 4\mathbf{i} + 2\mathbf{j} + 4\mathbf{k},
\]

and our normal vector is \( \mathbf{n} = <4,2,4> \). Then, picking \( P \) as the point we use to write the equation for the plane, the plane is

\[
4(x-2) + 2(y-1) + 4(z-1) = 0.
\]

(This is because we define a plane to be all points \( S = (x,y,z) \) for which the vector from \( P \) to \( S \) is perpendicular to \( \mathbf{n} \). Thus \( \mathbf{n} \cdot \overrightarrow{PS} = 0 \), which leads the indicated equation.) ♦