Problem 1. (5 pts.) Consider the function \( f(x, y) = x^2 y - 2x^2 - 2y^2 - 8y \). Find all critical points for \( f(x, y) \). If the discriminant for this function is \( D = -8(y-2) - 4x^2 \), determine the behavior the function has at each critical point. Be sure it is clear how you arrive at your conclusions.

Solution. Critical points occur when \( \nabla f = 0 \). Thus we need \( f_x = 0 \) and \( f_y = 0 \). Here, \( f_x = 2xy - 4x \) and \( f_y = x^2 - 4y - 8 \), so
\[
2x(y - 2) = 0 \quad \text{and} \quad x^2 - 4y - 8 = 0.
\]
The first requires \( x = 0 \) or \( y = 2 \). If \( x = 0 \), the second gives \( y = -2 \), and if \( y = 2 \) the second gives \( x = \pm 4 \). Thus our critical points are \((0, -2)\) and \((\pm 4, 2)\).

The discriminant tells us the behavior of the function at these points: if at a critical point \( D < 0 \), we have a saddle point. If \( D > 0 \) the second derivatives \( f_{xx} \) and \( f_{yy} \) have the same sign, indicating a maximum or minimum, which we can determine by looking at \( f_{xx} \) or \( f_{yy} \). Here \( D(\pm 4, 2) = -64 < 0 \), so \((\pm 4, 2)\) are saddle points. \( D(0, -2) = 32 > 0 \), so \((0, -2)\) is a maximum or minimum. \( f_{yy} = -4 \), so the function is “concave down,” and \((0, -2)\) is a maximum. (Note that we could also check \( f_{xx} \): \( f_{xx} = 2y - 4 \), which at \((0, -2)\) is also less than zero.)

Problem 2. (5 pts.) The level curves in the following figure are for some function \( f(x, y) \). Use the Midpoint Rule with \( n = 4 \) subregions to estimate the average value of the function on the region \( R = [0, 4] \times [0, 4] \). Be sure it is very clear how you obtain your estimate.

Solution. To use the Midpoint Rule, we break the region \( R \) into the four subregions outlined in the figure. We then estimate the value of the function in each region by using the function value at the midpoint (the points in the figure). These are: for the bottom left square, 6; for the bottom right square, \( \approx 5.7 \); for the top right, \( \approx 6.3 \); and for the top left, \( \approx 7 \). So \( \iint_R f(x, y) \, dA \approx 4(6 + 5.7 + 6.3 + 7) = 100 \). The average value of the function is this integral divided by the area of \( R \), or \( 100/16 = 6.25 \). This calculation isn’t nearly accurate to two decimal places, though.