Problem 1. (5 pts.) Consider the vector field \( \mathbf{F}(x, y) = y^2 \mathbf{i} - x \mathbf{j}. \) If \( C \) is the curve \( y = x^3 \) for \( 0 \leq x \leq 2 \), find \( \int_C \mathbf{F} \cdot d\mathbf{r}. \)

Solution. The curve \( C \) can be parameterized with \( r(t) = t \mathbf{i} + t^3 \mathbf{j}, \) \( 0 \leq t \leq 2. \) Then \( r'(t) = 1 \mathbf{i} + 3t^2 \mathbf{j}, \) so that \( \mathbf{F} \cdot dr = (y^2 - 3t^2x)dt = (t^6 - 3t^3)dt. \) Thus

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^2 (t^6 - 3t^3)dt = \left( \frac{1}{7}t^7 - \frac{3}{4}t^4 \right) \bigg|_{t=1}^{t=2} = \frac{1}{7}(2^7 - 1) - \frac{3}{4}(2^4 - 1) = \frac{127}{7} - \frac{15}{4}.
\]

\( \diamond \)

Problem 2. (5 pts.) Find a potential function for the conservative vector field \( \mathbf{F} = 9x^2 \cos(yz) \mathbf{i} - 3x^3z \sin(yz) \mathbf{j} - (3x^3y \sin(yz) - 2) \mathbf{k}. \) Then use it to find the integral \( \int_C \mathbf{F} \cdot d\mathbf{r} \) on the curve \( r(t) = t \mathbf{i} + (t - 2)^2 \mathbf{j} + (t^2 + 1) \mathbf{k} \) for \( 1 \leq t \leq 3. \)

Solution. The potential function \( f(x, y, z) \) satisfies \( \nabla f = \mathbf{F}, \) so \( f_x = 9x^2 \cos(yz). \) Integrating, \( f = 3x^3 \cos(yz) + g(y, z). \) Then \( f_y = -3x^3z \sin(yz) + g_y = -3x^3z \sin(yz), \) so \( g_y = 0 \) and \( g(y, z) = h(z). \) Finally, \( f_x = -3x^3y \sin(yz) + h' = -3x^3y \sin(yz) + 2, \) so \( h' = 2 \) and \( f(x, y, z) = 3x^3 \cos(yz) + 2z. \) The curve \( C \) extends from \( r(1) = <1, 1, 2> \) to \( r(3) = <3, 1, 10>, \) so

\[
\int_C \mathbf{F} \cdot d\mathbf{r} = f(r(3)) - f(r(1))
= f(3, 1, 10) - f(1, 1, 2)
= [3(3)^3 \cos(10) + 2(10)] - [3(1)^3 \cos(2) - 2(2)]
= 81 \cos(10) - 3 \cos(2) + 16.
\]

\( \diamond \)