Instructions Write on the front of your blue book your code letter from the posted list of participants. Do not write your name anywhere on your blue book. Each question is worth 10 points. For full credit, you must prove that your answers are correct even when the question doesn’t say “prove.” There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all; look for ones that seem easy and fun.

Problem 1. In an infinite checkerboard, each square has a natural number written in it. The number in any square is the average of the four numbers in the adjacent squares. Prove that all squares have the same number written in them.

Problem 2. Prove that every polynomial $p(x)$ can be expressed as the difference of two polynomials each of which is an increasing function of $x$.

Problem 3. You have a square kitchen which measures $2^n$ feet by $2^n$ feet. Think of it as a giant grid of squares one foot on a side. One of these squares is occupied by a (rather small) refrigerator. Show that, no matter which square this is, the rest of the kitchen can be tiled by tiles that look like

Problem 4. Consider a curved copy of the letter Y in 3-dimensional Euclidean space, i.e., the union of three closed arcs that are disjoint except for one point that is an endpoint of all three arcs. Prove that there is a plane that contains at least four points of this $Y$.

Problem 5. Prove that a positive integer $n$ is prime if and only if the equation

$$\frac{1}{x} - \frac{1}{y} = \frac{1}{n}$$

has a unique solution in positive integers $x, y$.

Problem 6. You are given that $f(x, y)$ is a real-valued function of two real variables and that, for each fixed real number $a$ both of the functions $f(a, y)$ and $f(x, a)$ are polynomials (of one variable each). Prove that $f(x, y)$ is a polynomial.

Problem 7. The six most active students in the department formed 30 committees, every two of which intersect, i.e., have at least one member in common. (No two committees have exactly the same members.) Prove that they can form one more committee that intersects each of the 30 existing committees.

Problem 8. Evaluate $\int_0^1 (\sqrt{1 - x^7} - \sqrt{1 - x^4}) \, dx$.

Problem 9. The homothety of the plane with center $P$ (a point of the plane) and ratio $r$ is defined to be the transformation sending any point $X$ of the plane to the unique point $Y$ such that the vector $\overrightarrow{PY}$ equals $r \cdot \overrightarrow{PX}$. Let $C$ be any closed, bounded, convex set in
the plane. Show that there is a homothety with ratio $-\frac{1}{2}$ (and some center) that maps $C$ into itself.

**Problem 10.** A regular $n$-gon is inscribed in a circle of radius 1. What is the product of the lengths of all its sides and all its diagonals?