Instructions

Write on the front of your blue book the code letter Laurel gave you. Do not write your name anywhere on your blue book. Each problem is worth the same amount. For full credit, you must prove that your answers are correct even when the question doesn’t say “prove.” There are lots of problems of widely varying difficulty. It is not expected that anyone will solve them all (or even nearly all); look for ones that seem easy and fun.

Problem 1. You are given a large rectangular array of cells (like a sheet of graph paper). Some of the cells are “marked” while the others are empty. You are also given two non-negative integers \(k\) and \(l\) (much smaller than the dimensions of the array of cells), and you mark new cells in accordance with the following rules. If a row contains at least \(k\) marked cells, then all other cells in that row are to be marked. If a column contains at least \(l\) marked cells, then all other cells in that column are to be marked. (Once a cell is marked, it remains so forever.) Suppose that, by repeated application of these rules, you can eventually mark all the cells. Prove that there must have been at least \(kl\) marked cells in the initial configuration.

Problem 2. \(a\) and \(b\) are elements in a (not necessarily commutative) ring with unit such that \(1 + ab\) is invertible. Prove that \(1 + ba\) is invertible.

Problem 3. Suppose \(a_1, \ldots, a_n\) are \(n\) complex numbers such that the power sums \(\sum_{i=1}^{n} a_i^k\) have the same value for \(k = 1, 2, \ldots, n + 1\). Prove that each \(a_i\) is 0 or 1.

Problem 4. 2000 students participated in a math competition. They had been arbitrarily assigned code numbers from 1 to 2000. All 2000 scores were different. Given that student 1 scored higher than students 2 through 1997, what is the probability that student 1 had the highest score overall?

Problem 5. Determine whether

\[
\frac{1}{(\ln N)^2} \sum_{n=1}^{N} (\sqrt[n]{n} - 1)
\]

converges as \(N \to \infty\) and, if it does, evaluate its limit.

Problem 6. If two of the three pairs of non-touching edges of a tetrahedron are perpendicular, prove that the third pair of non-touching edges is also perpendicular.

Problem 7. (a) A and B plan to play a game where they take turns tossing a coin until someone flips heads and thereby wins. On the basis of alphabetical order, A claims the right to go first, but B objects that this gives A an unfair advantage. To compensate, A offers to allow B to use a biased coin, whereas A will use a fair coin. Prove that the game is still biased in A’s favor.

(b) In view of your proof of part (a), B demands further adjustments in the probabilities of the individual coin flips. Because it is difficult to adjust the bias of a coin, the players agree to play the following variant of the game. A large (but finite) number of ping-pong balls are placed in a bag; some are labeled A and some B. The players alternately draw one ball out of the bag (each ball being equally likely and the draws being
independent). The first player to draw a ball labeled by his own name is the winner. Any ball drawn that doesn’t win this way is put back into the bag before the next draw. A will draw first, but to compensate for this advantage more balls will be labeled B than A. Is it possible to choose the number of balls with each label in such a way that the game is fair? If so, what should the numbers of balls be?

**Problem 8.** Into how many regions do \( n \) great circles in general position divide a sphere? (“General position” means that no three of the circles have a common point. “Regions” means two-dimensional regions, not parts of the circles themselves.)

**Problem 9.** Call a positive integer “lucky” if it differs by at most one from the sum of the squares of its digits. (“Digit” refers to the ordinary, base-10 notation.) For example, 1 is the only lucky number < 10. Find all other lucky numbers.

**Problem 10.** Consider configurations in which each square of an \( 8 \times 8 \) checkerboard is either empty or occupied by a checker. (The number of checkers can be anything from 0 to 64, and the color of the checkers doesn’t matter.) Let \( n \) be the number of configurations in which there is no X, i.e., no 5 checkers arranged like \( \circ \circ \circ \circ \) within a \( 3 \times 3 \) subsquare. (If more checkers are present at some of the 4 remaining positions in the \( 3 \times 3 \) subsquare, that still counts as an X.) Prove that \( n \) is a perfect square.