This first handout is an introduction to plane geometry. We will give rigorous definitions of points, lines, rays, line segments and angles. Various results seem evident and we will accept without proof. Such statements are called axioms. For example, we will simply assume that there goes a unique line through every two distinct points in the plane. Other results we will derive from our axioms. For example, we will prove that the sum of the angles in a triangle is equal to 180°.

One of the most important works in geometry, and in fact, in mathematics, is Euclid’s book “The Elements”. Euclid lived approximately from 325 BC until 265 BC. In “The Elements” he gives for the first time rigorous proofs of many results in geometry known to his contemporaries. Euclid starts with 5 postulates (axioms) and derives many propositions from these. An online version of the Elements can be found at: http://aleph0.clarku.edu/~djoyce/java/elements/elements.html

Euclid’s rigorous method has inspired scientists for millennia. We will not use the exact same axioms as Euclid, but our approach will be similar in style.

1. Lines

The plane \( \mathcal{E} \) is a set of points. Lines are certain subsets of the plane. So if \( P \) is a point in the plane an \( l \) is a line, then “\( P \) lies on \( l \)” or “\( l \) goes through \( P \)” really means that \( P \) is an element of the set \( l \).

**Axiom 1.** Through every two distinct points \( A, B \) in the plane goes a unique line.

![Diagram of Axiom 1](link)

This axiom is essentially Euclid’s first postulate. In other words, two distinct lines intersect in at most 1 point. Two distinct lines are called parallel if they do not intersect. We also consider any line parallel to itself. The unique line through \( A \) and \( B \) in Axiom 1 is denoted by \( AB \).

**Axiom 2.** Given a line \( l \) and a point \( A \), then there is a unique line through \( A \) which is parallel to \( l \).
This axiom is related to Euclid’s fifth postulate.

**Exercise 1.** Suppose that line \( l \) is parallel to \( m \) and line \( m \) is parallel to \( n \). Prove that \( l \) and \( n \) are parallel.

### 2. Rays, line segments, and in-betweenness

A point \( O \) on a line divides that line in 2 rays starting at \( O \). (The rays also contain \( O \).) The two rays are called each other’s *opposites*.

For every two distinct points \( O, A \) in the plane there is the unique line \( l = OA \) through \( O \) and \( A \). There are two rays along \( l \) that start at \( O \). Exactly one of them contains \( A \). This shows that there is a unique ray starting at \( O \) which goes through \( A \), which will be denoted by \( \uparrow OA \).

If \( A, B \) are two points on a line \( l \), then the line segment \( AB \) is the part of the line between \( A \) and \( B \) (including \( A \) and \( B \)). The line segment \( AB \) is the intersection of \( \uparrow AB \) and \( \uparrow BA \).

**Axiom 3.** Suppose that \( O, A, B \) are points on a line. Then the rays \( \uparrow OA, \uparrow OB \) are the same if and only if \( AB \) does not contain \( O \).

If \( A, B, C \) are distinct points on a line \( l \), then we say that \( B \) lies between \( A \) and \( C \) if the line segment \( AB \) contains \( C \).

**Proposition 1.** Suppose that \( A, B, C \) are distinct points on a line. Then exactly one of the following statements is true

1. \( A \) lies in between \( B \) and \( C \);
2. \( B \) lies in between \( C \) and \( A \);
3. \( C \) lies in between \( A \) and \( B \).
Proof. Suppose that \( \uparrow AB \) and \( \uparrow AC \) are distinct. Then \( BC \) contains \( A \) by Axiom 3. On the other hand \( \uparrow AC \) does not contain \( B \) because \( \uparrow AC \cap \uparrow AB = \{A\} \). So \( B \) does not lie on \( AC \). Similarly, \( AB \) does not contain \( C \). We have that (1) holds, but not (2) and (3).

Similarly, if \( \uparrow BC \neq \uparrow BA \) then (2) holds, but not (1) and (3). If \( \uparrow CA \neq \uparrow CB \) then (3) holds, but not (1) and (2).

Suppose that \( \uparrow AB = \uparrow AC \), \( \uparrow BC = \uparrow BA \) and \( \uparrow CA = \uparrow CB \). Then \( C \) lies on \( \uparrow AB \) and on \( \uparrow BA \) hence on \( BA \). This means that \( \uparrow CA \neq \uparrow CB \) by axiom 3 and we get a contradiction. \( \square \)

3. Two sides of a line

A line divides the plane into two regions. Suppose that \( l \) is a line. We say that two points \( A, B \) lie on the same side of \( l \) if the line segment \( AB \) does not intersect with \( l \). If \( AB \) intersects \( l \), then we say that \( A, B \) lie on opposite sides of line \( l \).

**Axiom 4.** Suppose that \( A, B, C \) are three points in a plane, and \( l \) is a line not equal to \( AB, BC \) or \( CA \). Then \( l \) intersects exactly 0 or 2 of the three line segments \( AB, BC, CA \).

**Proposition 2.** If \( A \) and \( B \) are on opposite sides of a line \( l \) and \( B \) and \( C \) are on the same side of \( l \), then \( A \) and \( C \) are on opposite sides of \( l \).

Proof. The line \( l \) intersects \( AB \), but not \( BC \). By Axiom 4 it should intersect \( CA \). So \( A \) and \( C \) lie on opposite sides of \( l \). \( \square \)

**Exercise 2.** Suppose that \( A \) and \( B \) lie on the same side of \( l \), and \( B \) and \( C \) lie on the same side of \( l \). Prove that \( A \) and \( C \) lie on the same side of \( l \).

**Exercise 3.** Suppose that \( A \) and \( B \) lie on opposite sides of \( l \) and \( B \) and \( C \) lie on opposite sides of \( l \). Then \( A \) and \( C \) lie on the same side of \( l \).

Two rays \( r \) and \( s \) with starting points \( A \) and \( B \) respectively, are called parallel, if the lines through them are parallel, and both rays lie on the same side of \( AB \).
4. ANGLES

To every two rays $r, s$ with the same starting point $A$ we associate an angle $\angle(r, s)$. Angles will be measured in degrees ($^\circ$). The angle $\angle(r, s)$ lies in the interval $[0^\circ, 180^\circ]$.

Suppose that $r, s, t$ are distinct rays with starting point $O$. We say that $s$ lies between $r$ and $t$ if there exists a point $P \neq O$ on $r$ and a point $Q \neq O$ on $t$ such that $PQ$ intersects $s$.

The following axiom essentially says that we have a reasonable measure for angles.

**Axiom 5.** Suppose that $r, s, t$ are rays with the same endpoint.

1. $\angle(r, r) = 0^\circ$,
2. $\angle(r, s) = \angle(s, r)$,
3. If $s$ lies between $r$ and $t$, then
   \[ \angle(r, t) = \angle(r, s) + \angle(s, t). \]
4. If $r$ and $s$ are opposite rays, then $\angle(r, s) = 180^\circ$.

Intuitively, angles can be measured in terms of the circumference of the circle, where the whole circle is $360^\circ$ and half a circle is $180^\circ$. By definition of $\pi$, the circumference
of the circle is exactly $2\pi$ times its radius. It is also common to measure angles in “radians”. So we have the following conversions

$$360^\circ = 2\pi \text{ rad}$$

or equivalently

$$1^\circ = \frac{\pi}{180} \text{ rad}.$$  

However, we will stick to degrees, because it is more convenient for our purposes.

Suppose that $r$ and $t$ are opposite rays starting at $O$. Then every other ray $s$ starting at $O$ lies between $r$ and $t$ and we have

$$180^\circ = \angle(r, t) = \angle(r, s) + \angle(s, t).$$

We call $\angle(r, s)$ and $\angle(s, t)$ supplementary angles.

**Exercise 4.** Suppose that $r_1, r_2$ are opposite rays with starting point $O$ and $s_1, s_2$ are opposite rays with the same starting point $O$. Prove that $\angle(r_1, s_1) = \angle(r_2, s_2)$. 
If \( O, A, B \) are distinct points in the plane with \( A \neq O \) and \( B \neq O \), then \( \angle AOB \) is defined as \( \angle (\uparrow OA, \uparrow OB) \).

The following axiom is again related to Euclid’s fifth postulate.

**Axiom 6.** Suppose that \( l \) and \( m \) are parallel lines, \( A, C \) are distinct points on \( l \), \( B, D \) are distinct points on \( m \). Let \( n \) be the line through \( A \) and \( B \) and assume that \( C \) and \( D \) are on opposite sides of the line \( n \). Then

\[
\angle (BAC) = \angle (ABD).
\]

**Theorem 3.** The sum of the angles inside a triangle is 180°. In other words, if \( A, B, C \) non-colinear points in the plane, then

\[
\angle (ABC) + \angle (BCA) + \angle (CAB) = 180^\circ.
\]
Proof. Let \( l \) be the line through \( C \) parallel to \( AB \). Choose points \( D, E \) on \( l \) such that \( D \) and \( B \) are on opposite sides of \( AC \), and \( A \) and \( E \) are on opposite sides of \( BC \). By Axiom 6, \( \angle CAB = \angle ACD \) and \( \angle ABC = \angle ECB \).

From the figure it is clear\(^1\) that \( BD \) intersects \( AC \), \( AE \) intersects \( BC \) and that \( D \) and \( E \) lie on opposite sides of \( C \).

We have \( \angle ECD = 180^\circ \), and

\[
180^\circ = \angle ECD = \angle ECB + \angle BCD
\]

Because \( BD \) intersects \( AC \) we have

\[
\angle BCD = \angle BCA + \angle ACD
\]

Finally we have

\[
180^\circ = \angle ECB + \angle BCA + \angle ACD = \angle ABC + \angle BCA + \angle CAB.
\]

\( \square \)

Exercise 5. Suppose that \( l \) and \( m \) are not parallel in Axiom 6. Show that \( \angle BAC \neq \angle ABD \).

A right angle is an angle equal to \( 90^\circ \). That is, it is equal to its supplementary angle. Three distinct points \( A, B, C \) in the plane, which do not lie on a line form a triangle. We denote this triangle by \( \triangle ABC \). We say that \( \triangle ABC \) is a right triangle (with the right angle at \( B \)) if \( \angle ABC = 90^\circ \). We say that \( \triangle ABC \) is a acute triangle if all angles (\( \angle ABC, \angle BCA, \angle CAB \)) are smaller than \( 90^\circ \). A triangle is called obtuse if one of the angles is larger than \( 90^\circ \). Every triangle is either a right, a acute or an obtuse triangle.

\(^1\)Pictures can be quite misleading. Deducing results from a picture should be kept to a minimum.
Sometimes we would like to measure angles beyond 180°. For this we use oriented angles. For any two rays $r, s$ starting at the same point $O$ we define $\angle(r, s)$. We use the notation $\angle BOA = \angle(\uparrow OB, \uparrow OA)$. We have the following rules:

1. If $B, O, A$ appear counterclockwise (seen from the inside of $\triangle BOA$) then $\angle BOA = \angle BOA$.
2. If $B, O, A$ appear clockwise, then $\angle BOA = 360° - \angle BOA$.
3. $0° \leq \angle(r, s) < 360°$ and $\angle(r, s) = 0$ if and only if $r = s$.
4. If $r, s, t$ are rays starting from $O$, then either
   (a) $\angle(r, t) = \angle(r, s) + \angle(s, t)$ or
   (b) $\angle(r, t) = \angle(r, s) + \angle(s, t) - 360°$.

Note that

$\angle(r, s) + \angle(s, r) = 360°$.

unless $r$ and $s$ are the same (in which case the sum of the two angles is just $0°$). For example, in the figure below, we have $\angle(r, s) = 90°$, but $\angle(s, r) = 270°$.

If $A, B, C$ are points of a triangle, lie counter clockwise, then

$\angle ABC + \angle BCA + \angle CAB = 180°$.

However, if $A, B, C$ are clockwise, then

$\angle ABC + \angle BCA + \angle CAB = (360° - \angle CBA) + (360° - \angle ACB) + (360° - \angle BAC) = 1080° - 180° = 900°$.

5. POLYGONS

A polygon or $k$-gon is a sequence of points $A_1, A_2, \ldots, A_k$ simply denoted by $A_1A_2 \cdots A_k$. 
The sides of the polygon \( A_1 A_2 \cdots A_k \) are
\[ A_1 A_2, A_2 A_3, \ldots, A_{k-1} A_k, A_k A_1. \]

The angles is the polygon are
\[ \angle A_k A_1 A_2, \angle A_1 A_2 A_3, \angle A_2 A_3 A_4, \ldots, \angle A_{k-2} A_{k-1} A_k, \angle A_{k-1} A_k A_1. \]

Its oriented angles are
\[ \angle A_k A_1 A_2, \angle A_1 A_2 A_3, \angle A_2 A_3 A_4, \ldots, \angle A_{k-2} A_{k-1} A_k, \angle A_{k-1} A_k A_1. \]

For a \( k \)-gon we use the convention that
\[ A_{k+1} = A_1, A_{k+2} = A_2, \ldots \]

and also
\[ A_0 = A_k, A_{-1} = A_{k-1}, \ldots. \]

The polygon is called \textit{convex} if the ray \( \uparrow A_i A_j \) lies between \( \uparrow A_i A_{i-1} \) and \( \uparrow A_i A_{i+1} \) for all \( i \) and \( j \) with 1 \( \leq i, j \leq k \) and \( j \neq i-1, i, i+1 \).

\textbf{Theorem 4.} \textit{The sum of the angles of a convex \( k \)-gon is} \( k \cdot 180^\circ - 360^\circ \).
Proof. Connect $A_1$ with the other points $A_2, A_3, \ldots, A_k$. The sum of all angles inside the $k$-gon is equal to the sum of all angles in $k - 2$ triangles, hence equal to

$$(k - 2) \cdot 180^\circ = k \cdot 180^\circ - 360^\circ.$$ 

\square

Exercise 6. Suppose that $A_1A_2A_3A_4A_5$ is a convex pentagon. What is

$$\angle A_2A_4A_1 + \angle A_3A_5A_2 + \angle A_4A_1A_3 + \angle A_5A_2A_4 + \angle A_1A_3A_5?$$

Exercise 7. Prove that among 4 distinct points in the plane, one can choose 3 points which lie on a line or form a right or obtuse triangle.

We can generalize Theorem 4 this to non-convex polygons.

**Theorem 5.** Assume that a polygon $A_1A_2 \cdots A_k$ does not intersect itself, and that the vertices of the polygon are labeled counterclockwise. Then the sum of the oriented angles is equal to $k \cdot 180^\circ - 360^\circ$. 