PROBLEM SET 1: INDUCTION (DUE 9/12/2002)

HARM DERksen

Let $\mathbb{N} = \{1, 2, \cdots \}$ be the natural numbers. Our convention here is that the
natural numbers start with 1. One important property of $\mathbb{N}$ is the induction
principle. It is in fact rather an axiom than a theorem, i.e., we simply assume
that the induction principle for natural numbers is true.

**Axiom 1.** Suppose we have a statement which depends on a natural number
$n \in \mathbb{N}$. Suppose that

1. the statement is true for $n = 1$, and
2. if for a natural number $m$, the statement is true for $n = m$, then the state-
   ment is also true for $n = m + 1$.

Then the statement is true for all natural numbers $n$.

**Example 1.** Show that

$$1 + 2 + \cdots + n = \frac{n(n + 1)}{2}. \tag{1}$$

for all $n \in \mathbb{N}$.

*Proof.* This problem screams for induction. Indeed, induction seems the right
tool here, because we have to prove a formula for all natural numbers $n$.

We check (1) for $n = 1$:

$$1 = \frac{1(1 + 1)}{2}.$$

Suppose that (1) is true for $n = m$, i.e.,

$$1 + 2 + \cdots + m = \frac{m(m + 1)}{2}.$$

then

$$1 + 2 + \cdots + m + (m + 1) = \frac{m(m + 1)}{2} + (m + 1) = \frac{(m + 1)(m + 2)}{2}$$

so (1) is also true for $n = m + 1$. Using the induction principle, (1) holds for all
natural numbers $n \in \mathbb{N}$. \hfill $\square$

Equivalent to induction is the following axiom which is sometimes called *strong
induction*:
**Axiom 2.** Suppose we have a statement which depends on a natural number \( n \in \mathbb{N} \). Suppose that whenever \( m \in \mathbb{N} \) is a natural number and for all \( n \in \mathbb{N} \), \( n < m \) the statement is true then the statement is true for \( n = m \). Then the statement is true for all \( n \in \mathbb{N} \).

**Example 2.** Recall that a prime number is a natural number with exactly 2 divisors. Every natural number \( \geq 2 \) is a product of prime numbers.

**Proof.** We use strong induction to prove that every natural number \( n \geq 2 \) is a product of prime numbers. Suppose \( n \geq 2 \) is a natural number. If \( n \) is a prime number then we are done. Otherwise we can write \( n = ab \) with \( a, b \) natural numbers and \( a, b < n \). By induction we know that \( a \) and \( b \) are both products of prime numbers. Therefore also \( n \) is a product of prime numbers. By strong induction, all natural numbers \( n \in \mathbb{N} \) are products of prime numbers. \( \square \)

Also equivalent to the induction axiom is the following axiom.

**Axiom 3.** Every nonempty subset \( S \) of \( \mathbb{N} \) has a smallest element.

The three axioms are all equivalent. The proof is an exercise in logical reasoning.

**Theorem 1.** The axioms 1, 2 and 3 are equivalent.

**Axiom 1 \( \Rightarrow \) Axiom 2.** Suppose that \( P(n) \) is a statement for all \( n \in \mathbb{N} \), and whenever \( P(n) \) holds for all \( n < m \), then \( P(m) \) holds. Let \( Q(m) \) be the statement "\( P(n) \) holds for all \( n < m \)". Now \( Q(1) \) is an empty statement since there are no natural numbers smaller than 1. In particular \( Q(1) \) is true. If \( Q(m) \) holds, then \( P(n) \) holds for all \( n < m \). But then \( P(m) \) holds and therefore \( Q(m + 1) \) holds. By induction, \( Q(m) \) holds for all natural numbers \( m \in \mathbb{N} \). It follows that \( P(n) \) holds for all natural numbers \( n \in \mathbb{N} \). \( \square \)

**Axiom 2 \( \Rightarrow \) Axiom 3.** Suppose that \( S \) does not have a smallest element. By strong induction on \( n \) we will show that \( n \notin S \) for all \( n \in \mathbb{N} \). Let us assume that \( n \notin S \) for all \( n < m \). If \( m \in S \), then \( m \) would be the smallest element of \( S \). Therefore \( m \notin S \). By strong induction we have that for any natural number \( n \), \( n \notin S \). In other words \( S \) must be the empty set which contradicts the assumptions on \( S \). We conclude that \( S \) must have a smallest element. \( \square \)

**Axiom 3 \( \Rightarrow \) Axiom 1.** Suppose that \( P(n) \) is a statement for all \( n \in \mathbb{N} \). Also we assume that \( P(1) \) holds and whenever \( P(n) \) holds, then \( P(n + 1) \) holds as well. Let

\[ S = \{ n \in \mathbb{N} \mid P(n) \text{ does not hold} \}. \]

Suppose that \( S \) is nonempty. Then \( S \) has a smallest element, say \( n \). Now \( n \) cannot be 1 since \( P(1) \) holds. Now \( n - 1 \) is also an integer and \( P(n - 1) \) must hold since \( n - 1 < n \) and \( n \) was the smallest element of \( S \). But \( P(n - 1) \) implies
$P(n)$ which contradicts $n \in S$. Therefore $S$ must be empty and $P(n)$ must hold for all $n \in \mathbb{N}$.

PROBLEMS

Problem 1.
(a) * Prove that
\[
1^2 + 2^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6}
\]
for all $n \in \mathbb{N}$.
(b) * Show that
\[
1^3 + 2^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2
\]
for all $n \in \mathbb{N}$.
(c) *** Can you find and prove a formula for
\[
1^4 + 2^4 + \cdots + n^4
\]
(d) ***** Can you find and prove a formula for
\[
1^k + 2^k + \cdots + n^k
\]
for every natural number $k$?

Problem 2.
(a) * Show that
\[
1 + x + x^2 + \cdots + x^n = \frac{x^{n+1} - 1}{x - 1}.
\]
(b) ** Find and prove a formula for
\[
1 + 2x + 3x^2 + \cdots + nx^{n-1}.
\]
(c) *** What happens in (a) and (b) if we take the limit $x \to 1$?
(d) ***** Find and prove a formula for
\[
1 + 2^k x + 3^k x^2 + \cdots + n^k x^{n-1}
\]

Problem 3. ****
(a). ((UM)$^2$C5, 5) For $n \geq 1$, let $S_n$ be the set of points $(x, y)$ in the plane with integral coordinates satisfying $x \geq 0$, $y \geq 0$ and $x + y \leq n$. Prove that $S_n$ is not contained in the union of $n$ straight lines.
(b). How many possibilities are there for the line configurations with exactly $n$ straight lines such that exactly one point is missed?

Problem 4. ** Show that
\[
1 - \frac{1}{2} + \frac{1}{3} - \cdots + \frac{1}{2n-1} - \frac{1}{2n} = \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{2n}
\]
for all $n \in \mathbb{N}$.
Problem 5. ***** Let \( \mathbb{N}^d \) be the set of \( d \)-tuples of natural numbers. We write \( a \leq b \) for two elements \( a = (a_1, a_2, \ldots, a_d), b = (b_1, b_2, \ldots, b_d) \in \mathbb{N}^d \) if \( a_i \leq b_i \) for \( i = 1, 2, \ldots, d \). Suppose that \( S \subseteq \mathbb{N}^d \) is a nonempty subset. An element \( a \in S \) is called minimal if \( b \in S, b \leq a \) implies \( b = a \). Show that \( S \) can only have finitely many minimal elements. (This problem is known as Dixon’s Lemma. It can be used to prove that ideals in the polynomial ring \( K[x_1, \ldots, x_n] \) over a field \( K \) are finitely generated.)

Problem 6 (MMPC 29, 1). *** Sometimes one finds in an old park a tetrahedral pile of cannon balls, that is, a pile each layer of which is a tightly packed triangular layer of balls.

(a). How many cannon balls are in a tetrahedral pile of cannon balls of \( N \) layers?

(b). How high is a tetrahedral pile of cannon balls of \( N \) layers? (Assume each cannon ball is a sphere of radius \( R \).)