PROBLEM SET 4: INEQUALITIES
(DUE 10/3/2002)

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This problem set is about inequalities. In fact, it contains everything I know about inequalities. (Hopefully it is not too overwhelming.) First we will consider the inequality $x^2 \geq 0$ for real numbers $x$. This very simple inequality can be applied to prove a wide range of inequalities.

A very powerful method of proving inequalities is the use of convex and concave functions. We will use this to prove an important and very useful inequality

“harmonic average” ≤ “geometric average” ≤ “arithmetic average”.

Finally we also will discuss the Schwarz inequality which is important in geometry.

When encountered with a new inequality to prove, one can try to deduce it from one of the inequalities one already knows.

1. SOME ELEMENTARY INEQUALITIES

Let us start with one of the most fundamental inequality for real numbers, namely

$$x^2 \geq 0$$

for all $x \in \mathbb{R}$. Starting with this inequality you can actually prove many other inequalities. For example, if $a, b \in \mathbb{R}$ then we have

$$(a - b)^2 \geq 0 \iff a^2 - 2ab + b^2 \geq 0 \iff a^2 + b^2 \geq 2ab \iff \frac{a^2 + b^2}{2} \geq ab.$$  

If $x,y \geq 0$ then by taking $a = \sqrt{x}$ and $b = \sqrt{y}$ we get

$$\frac{x + y}{2} \geq \sqrt{xy}.$$ 

Also by using

$$(a - b)^2 + (b - c)^2 + (c - a)^2 \geq 0$$

and writing this out we see that

$$2a^2 + 2b^2 + 2c^2 \geq 2ab + 2bc + 2ca$$

so

$$a^2 + b^2 + c^2 \geq ab + bc + ca.$$ 

Sometimes you can use the conditions on the variables directly to deduce the inequality, as the following example shows:
Example 1. * If $0 \leq x, y, z \leq 1$ then
\[
1 + xy + yz + zx \geq x + y + z \geq xy + yz + zx
\]

Proof. Since $1 - x, 1 - y, 1 - z$ are nonnegative we have
\[
(1 - x)(1 - y)(1 - z) \geq 0
\]
and working this out gives
\[
1 + xy + yz + zx \geq x + y + z + xyz \geq x + y + z.
\]
Also, since
\[
x(1 - y) + y(1 - z) + z(1 - x) \geq 0
\]
we get the other inequality as well. \qed

2. **Convexity**

Let $f$ be a real-valued function on an interval $I \subseteq \mathbb{R}$. Now $f$ is said to be convex if
\[
f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b)
\]
for all $t \in [0, 1]$ and all $a, b \in I$ (the chord between $(a, f(a))$ and $(b, f(b))$ lies above the graph of $f$). The function $f$ is said to be concave if
\[
f(ta + (1 - t)b) \geq tf(a) + (1 - t)f(b)
\]
for all $t \in [0, 1]$ and all $a, b \in I$ (the chord between $(a, f(a))$ and $(b, f(b))$ lies below the graph of $f$).

**Theorem 1.** Suppose that $f$ is a real-valued function on $I \subseteq \mathbb{R}$, $x_1, x_2, \ldots, x_n \in I$, and $t_1, t_2, \ldots, t_n \in [0, 1]$ with $t_1 + t_2 + \cdots + t_n = 1$. If $f$ is convex, then
\[
f(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n).
\]
If $f$ is concave, then
\[
f(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \geq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n).
\]

Proof. Suppose that $f$ is convex. We will prove the statement by induction on $n$, the case $n = 1$ being trivial. Suppose that we already have proven that
\[
f(t_1 x_1 + t_2 x_2 + \cdots + t_n x_n) \leq t_1 f(x_1) + t_2 f(x_2) + \cdots + t_n f(x_n).
\]
for all $x_1, x_2, \ldots, x_n \in I$ and all $t_1, t_2, \ldots, t_n \in [0, 1]$ with $t_1 + t_2 + \cdots + t_n = 1$.

Suppose now that $x_1, x_2, \ldots, x_{n+1} \in I$ and $t_1, \ldots, t_{n+1} \in [0, 1]$ with $t_1 + t_2 + \cdots + t_{n+1} = 1$. Define $s_i = t_i / (1-t_{n+1})$ for $i = 1, 2, \ldots, n$. Note that $s_1 + s_2 + \cdots + s_n = 1$. Take $a = s_1 x_1 + s_2 x_2 + \cdots + s_n x_n$, $b = x_{n+1}$ and $t = 1-t_{n+1}$. From the definition of convexity and the induction hypothesis follows that
\[
f(t_1 x_1 + \cdots + t_{n+1} x_{n+1}) = f(ta + (1 - t)b) \leq tf(a) + (1 - t)f(b) =
\]
\[
= (1 - t_{n+1})f(s_1 x_1 + \cdots + s_n x_n) + t_{n+1} f(x_{n+1}) \leq
\]
\[
\leq (1 - t_{n+1})(s_1 f(x_1) + s_2 f(x_2) + \cdots + s_n f(x_n)) + t_{n+1} f(x_{n+1}) =
\]
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\[ = t_1 f(x_1) + \cdots + t_{n+1} f(x_{n+1}). \]

To prove the second statement, observe that \( f \) is concave if and only if \( -f \) is convex. Then apply the first statement to \( -f \). \( \square \)

In particular the case \( t_1 = t_2 = \cdots = t_n = 1/n \) is interesting.

**Corollary 1.** If \( f \) is convex on \( I \), then

\[
\frac{f(x_1) + \cdots + f(x_n)}{n} \leq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)
\]

for all \( x_1, \ldots, x_n \in I \).

If \( f \) is concave on \( I \), then

\[
\frac{f(x_1) + \cdots + f(x_n)}{n} \geq f\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right)
\]

for all \( x_1, \ldots, x_n \in I \).

**Theorem 2.** Suppose that \( f \) is a real-valued function on an interval \( I \subseteq \mathbb{R} \) with a second derivative. If \( f''(x) \geq 0 \) for all \( x \in I \), then \( f \) is convex. If \( f''(x) \leq 0 \) for all \( x \in I \), then \( f \) is concave. (The converse of these statements are also true).

**Proof.** If \( f''(x) \geq 0 \) for all \( x \in I \) then \( f'(x) \) is weakly increasing on the interval \( I \). Suppose that \( a, b \in I \) and \( t \in [0, 1] \). Define \( c = ta + (1-t)b \). By the Mean Value Theorem, there exist \( \alpha \in (a, c) \) and \( \beta \in (c, b) \) such that

\[
f'(\alpha) = \frac{f(c) - f(a)}{c - a} \quad \text{and} \quad f'(\beta) = \frac{f(b) - f(c)}{b - c}.
\]

Since \( \alpha < \beta \) and \( f' \) is weakly increasing, we have

\[
\frac{f(ta + (1-t)b) - f(a)}{(1-t)(b-a)} = \frac{f(c) - f(a)}{c - a} = f'(\alpha) \leq f'(\beta) = \frac{f(b) - f(c)}{b - c} = \frac{f(b) - f(ta + (1-t)b)}{t(b-a)}
\]

Multiplying out gives

\[
f(ta + (1-t)b) \leq tf(a) + (1-t)f(b).
\]

This shows that \( f \) is convex.

The second statement follows from the first statement, applied to \( -f \). \( \square \)
3. Arithmetics, Geometric and Harmonic Mean

**Theorem 3.** Let \(x_1, x_2, x_3, \ldots, x_n > 0\). We define the Arithmetic Mean by

\[
A(x_1, x_2, \ldots, x_n) = \frac{x_1 + x_2 + \cdots + x_n}{n},
\]

the Geometric Mean by

\[
G(x_1, x_2, \ldots, x_n) = \sqrt[n]{x_1 x_2 \cdots x_n}
\]

and the Harmonic Mean by

\[
H(x_1, x_2, \ldots, x_n) = \frac{n}{\frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_n}}.
\]

Then we have

\[
H(x_1, \ldots, x_n) \leq G(x_1, \ldots, x_n) \leq A(x_1, \ldots, x_n).
\]

**Proof.** Let \(f(x) = \log(x)\). Then \(f''(x) = -1/x^2 < 0\) for \(x > 0\) so \(f\) is concave on the interval \((0, \infty)\). It follows that

\[
\log\left(\frac{x_1 + x_2 + \cdots + x_n}{n}\right) \geq \frac{\log(x_1) + \log(x_2) + \cdots + \log(x_n)}{n}.
\]

Applying the exponential function (which is an increasing function) to both sides yields

\[
\frac{x_1 + x_2 + \cdots + x_n}{n} \geq \sqrt[n]{x_1 x_2 \cdots x_n}.
\]

If we now take \(y_i = \frac{1}{x_i}\) then we get

\[
\frac{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}}{n} \geq \frac{1}{\sqrt[n]{y_1 y_2 \cdots y_n}}.
\]

Taking the reciprocal yields

\[
\frac{n}{\frac{1}{y_1} + \frac{1}{y_2} + \cdots + \frac{1}{y_n}} \leq \sqrt[n]{y_1 y_2 \cdots y_n}.
\]

\(\square\)

4. The Schwarz Inequality

Another important inequality is the Schwarz inequality. For vectors \(x = (x_1, \ldots, x_n)\) and \(y = (y_1, \ldots, y_n)\) in \(\mathbb{R}^n\) one defines

\[
x \cdot y = x_1 y_1 + \cdots + x_n y_n.
\]

Note that \(x \cdot y = y \cdot x\), \((x + y) \cdot z = x \cdot z + y \cdot z\) and \((tx) \cdot y = t(x \cdot y)\) for \(t \in \mathbb{R}\) and \(x, y, z \in \mathbb{R}^n\).

The norm of the vector \(x\) is defined by

\[
\|x\| = \sqrt{x \cdot x} = \sqrt{x_1^2 + \cdots + x_n^2}.
\]
Theorem 4. Suppose that $x = (x_1, x_2, \ldots, x_n)$, $y = (y_1, y_2, \ldots, y_n) \in \mathbb{R}^n$, then

$$|x_1 y_1 + \cdots + x_n y_n| \leq \sqrt{x_1^2 + \cdots + x_n^2} \sqrt{y_1^2 + \cdots + y_n^2}$$

or in short form:

$$|x \cdot y| \leq \|x\|\|y\|.$$  

Proof. For any vector $a \cdot a \geq 0$. In particular, if we take $a = x + ty$ we get

$$(x + ty) \cdot (x + ty) = x \cdot x + 2t(x \cdot y) + t^2(y \cdot y) \geq 0$$

for all $t \geq 0$. Viewed as a quadratic polynomial in $t$, this polynomial has a nonpositive discriminant. The discriminant is

$$4(x \cdot y)^2 - 4(x \cdot x)(y \cdot y) \leq 0$$

In particular we have

$$(x \cdot y)^2 \leq (x \cdot x)(y \cdot y)$$

and taking square roots gives us

$$|x \cdot y| \leq \sqrt{x \cdot x} \sqrt{y \cdot y} = \|x\|\|y\|.$$  

The Schwarz inequality is important in Euclidean geometry in dimension 2, 3 or higher. In particular, one often defines the angle $\phi$ between two vectors $x, y$ by

$$\cos(\varphi) = \frac{x \cdot y}{\|x\|\|y\|}, \quad 0 \leq \varphi \leq \pi.$$  

The Schwarz inequality tells us that this definition makes sense, since the right-hand side has absolute value at most 1.

Another famous geometric inequality is the triangle inequality. If $a, b, c$ are the lengths of the sides of a triangle, then $a + b \geq c$ (and also $a + c \geq b$ and $b + c \geq a$).

5. PROBLEMS

Problem 1. ** Use the inequality $\frac{x+y+w}{2} \geq \sqrt{xyz}$ repeatedly to prove

$$\frac{x+y+z+w}{4} \geq \frac{4\sqrt{xyzw}}{4}$$

for all $x, y, z, w \geq 0$.

Problem 2. ** Prove that

$$x_1^2 + x_2^2 + \cdots + x_n^2 \geq \frac{2}{n-1} \sum_{1 \leq i < j \leq n} x_i x_j$$

for all positive integers $n$.

Problem 3. ** For positive real $a, b, c$ prove that

$$b^3c^3 + c^3a^3 + a^3b^3 \geq 3a^2b^2c^2.$$
Problem 4. ** For positive $x_1, x_2, \ldots, x_n$, prove that
\[
\frac{x_1}{x_2} + \frac{x_2}{x_3} + \cdots + \frac{x_{n-1}}{x_n} + \frac{x_n}{x_1} \geq n.
\]

Problem 5. ** If $x \leq y \leq z$ and $y > 0$, prove that
\[
x + z - y \geq \frac{xz}{y}
\]

Problem 6. *** For nonnegative real $u_1, \ldots, u_n$, prove that
\[
\left( \sum_{i=1}^{n} u_i \right)^3 \leq n^2 \sum_{i=1}^{n} u_i^3.
\]
(Use that $x^3$ is convex for $x \geq 0$).

Problem 7. ** Let
\[
s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}.
\]
Prove that
\[
n((n+1)^n - 1) \leq s_n \leq n - \frac{n-1}{n^{1/(n-1)}}.
\]
(Hint: use the geometric and arithmetic mean for $1 + 1 + \frac{1}{2}, \ldots, 1 + \frac{1}{n}$ and for $1 - \frac{1}{2}, 1 - \frac{1}{3}, \ldots, 1 - \frac{1}{n}$.)

Problem 8. **** Prove if $a, b, c > 0$ then $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$.

Problem 9. *** Let $p_1, p_2, \ldots, p_n$ be any $n$ points on the sphere
\[
\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}.
\]
Prove that the sum of the squares of the distances between them is at most $n^2$.

Problem 10. * Let $Q$ be a convex quadrilateral (i.e., the diagonals lie inside the figure). Let $S$ be the sum of the lengths of the diagonals and let $P$ be the perimeter. Prove
\[
\frac{1}{2} P < S < P.
\]

Problem 11. *** Prove that
\[
\frac{1}{2} \cdot \frac{3}{4} \cdots \frac{999,999}{1,000,000} < \frac{1}{1,000}.
\]

Problem 12. * Prove that
\[
\frac{x_1}{x_1 + x_2} + \frac{x_2}{x_2 + x_3} + \cdots + \frac{x_{n-1}}{x_{n-1} + x_n} + \frac{x_n}{x_n + x_1} \geq 1
\]

Problem 13 (Putnam 1988, B2). *** Prove or disprove: If $x$ and $y$ are real numbers with $y \geq 0$ and $y(y+1) \leq (x+1)^2$, then $y(y-1) \leq x^2$. 

Problem 14 (IMO 1995, 2). ***** Let $a, b, c$ be positive real numbers such that $abc = 1$. Prove that
\[ \frac{1}{a^3(b + c)} + \frac{1}{b^3(c + a)} + \frac{1}{c^3(a + b)} \geq \frac{3}{2}. \]

Problem 15. ***** Suppose that $x_1, x_2, \ldots, x_n$ are positive real numbers. Prove that
\[ \frac{x_1}{x_2 + x_3} + \frac{x_2}{x_3 + x_4} + \cdots + \frac{x_n}{x_1 + x_2} \geq \frac{n}{4} \]
(indices go cyclic).

Problem 16. **** Prove the Hölder inequality: If $1/p + 1/q = 1$ and $x, y \in \mathbb{R}^n$ then
\[ |x \cdot y| \leq \|x\|_p \|y\|_q \]
where $\|x\|_p = (|x_1|^p + |x_2|^p + \cdots + |x_n|^p)^{1/p}$. (Hint: Use that $\log(x)$ is convex and prove $x_i y_i \leq x_i^p / p + y_i^q / q$. Then prove the inequality in the special case that $\|x\|_p = \|y\|_q = 1$. Reduce the general case to this special case.)