PROBLEM SET 7: THE INVARIANCE PRINCIPLE (DUE 10/24/2002)

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Vaguely speaking, an invariant is a quantity that remains the same after certain operations. For many problems it is useful to identify such invariants.

Example 1. ** We cut out two opposite corner fields of a chessboard. Is it possible to put 31 domino tiles (of size 2 \times 1) on the remaining 62 fields of the chessboard?

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THE INVARIANCE PRINCIPLE

Example 2. **** Suppose that $p$, $q$ and $n$ are positive integers such that $n$ is not divisible by $p$ or by $q$. Prove that an $n \times n$ floor cannot be tiled by $p \times p$ or by $q \times q$ tiles.

Proof. For any integer $k$, let $\zeta_k = e^{2\pi i/k}$ be the $k$-th primitive root of unity. Observe that

$$1 + \zeta + \zeta^2 + \cdots + \zeta^{l-1} = (1 - \zeta^l)/(1 - \zeta)$$

is equal to 0 if and only if $l$ is divisible by $k$. (For this problem, it is useful to have seen some complex numbers and to know a little bit about roots of unity.) Fill the $n \times n = n^2$ fields with complex numbers as follows. Number the rows and columns by $0, 1, 2, \ldots, n-1$. Put the complex number $\zeta_p^i \zeta_q^j$ on the field in row $i$ and column $j$. The sum of all numbers over all fields is

$$\sum_{0 \leq i,j < n} \zeta_p^i \zeta_q^j = (1 + \zeta_p + \cdots + \zeta_p^{n-1})(1 + \zeta_q + \cdots + \zeta_q^{n-1})$$

is nonzero since $p$ and $q$ do not divide $n$. On the other hand, if we place a $p \times p$ tile (with one corner at $(k, l)$), then the sum of all complex numbers under the tile is

$$\sum_{i=k}^{k+p-1} \sum_{j=l}^{l+p-1} \zeta_p^i \zeta_q^j = \zeta_p^{2k} \sum_{i=0}^{p-1} \zeta_p^i \left( \sum_{i=0}^{p-1} \zeta_q^i \right) = 0$$

since

$$\sum_{i=0}^{p-1} \zeta_p^i = 0.$$

Similarly all the numbers under a $q \times q$ tile sum up to 0. This shows that it is not possible to tile the $n \times n$ floor with $p \times p$ and $q \times q$ tiles. ☐

PROBLEMS

Problem 1. ** Show that a $10 \times 10$ rectangle cannot be divided up into 25 $4 \times 1$ rectangles. (Hint: Color the 100 $1 \times 1$ fields with four colors in a suitable way, and use a similar argument as for the chessboard/domino puzzle.)

Problem 2. *** Show that $12 \times 11$ rectangular floor cannot be covered using only tiles of the following shapes:

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\[ \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \quad \begin{array}{ccc}
1 & 1 & 1 \\
1 & 1 & 1 \\
1 & 1 & 1 \\
\end{array} \]
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Problem 3. ** Cut out one corner of a $2^n \times 2^n$ chess board. Show that it is to cover the chess board with $(4^n - 1)/3$ L-shaped pieces? The case $n = 2$ is shown below. (You do not need the invariance principle here.)
Problem 4. **** Suppose that $p$, $q$ and $r$ are distinct prime numbers and $N > 2pqr$. Show that an $N \times N$ floor can be tiled with $p \times p$, $q \times q$ and $r \times r$ tiles. (Hint: Write $N = apq + bpr + cqr$ for certain nonnegative integers $a, b, c$. Use this to divide the $N \times N$ floor in regions which are easy to tile.)

Problem 5. **** A rectangle $R$ is divided into smaller rectangles. Each of the smaller rectangles has at least one side, whose length is an integer. Show that $R$ itself has at least one side which is an integer.

Problem 6. ** You are at the coordinates $(1,0,0)$ in $\mathbb{R}^3$ where we use the usual $xyz$ coordinate axis. A three dimensional knight jump is if you move $\pm 1$ along one coordinate axis, $\pm 2$ along a second coordinate axis and $\pm 3$ along the third coordinate axis. For example one could jump from $(1,0,0)$ to $(1,0,0) + (2,-1,3) = (3,-1,3)$. Then one could jump to $(3,-1,3) + (-3,2,-1) = (0,1,2)$ and from there to $(0,1,2) + (1,2,-3) = (1,3,-1)$. Show that it is impossible to land at $(0,0,0)$ after finitely many three dimensional knight jumps.

Problem 7. ** There are 25 matches on the table. Two players take turns. Each turn they have to take away 1,2 or 3 matches. The person taking the last match loses. Show that the second player always can win this game. (Try it first with 5,9 and 13 matches instead.)

![Match Arrangement](http://www.geocities.com/TimesSquare/4934/javamatc.html)

(You can play it online at http://www.geocities.com/TimesSquare/4934/javamatc.html, http://www.safe4kids.org/kids/5matches.htm)

Problem 8 (IMO). **** To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers $x, y, z$ respectively and $y < 0$ then the following operation is allowed: the numbers $x, y, z$ are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.
Problem 9. **** Define a sequence $x_1, x_2, x_3, \ldots$ by $x_1 = 1, x_2 = 5$ and

$$x_{n+1} = \frac{x_n}{2} + x_{n-1} - \frac{x_{n-1}^2}{2x_n}$$

for $n \geq 2$. What is $\lim_{n \to \infty} x_n$?

Problem 10. *** We start with the numbers

$$1, 2, 3, 4, 5, 6, 7, 8, 9, 10$$

Then we replace two numbers, say $x$ and $y$, by $xy/(x + y)$. We repeat this until there is only one number left. Show that, regardless how you do it, this number is always equal to $2520/7381$. (For example, we could replace 3 and 6 by $3 \cdot 6/(3 + 6) = 2$ to get the sequence

$$1, 2, 4, 5, 7, 8, 9, 10, 2.$$  

Then we can replace 9 and 10 by $9 \cdot 10/(9 + 10) = 90/19$ to get the sequence

$$1, 2, 4, 5, 7, 8, 2, 90/19.$$  

etc.)

Problem 11 (after a well-known puzzle). **** In a $4 \times 4$ square, we put the numbers 2, 1, 3, 4, 5, 6, …, 16 (see below). In each move, we may exchange 16 with one of its neighbors (neighbor means sharing an edge). Is it possible to get 1, 2, 3, …, 16 after finitely many moves (see second picture).

\[
\begin{array}{cccc}
2 & 1 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]

\[
\begin{array}{cccc}
1 & 2 & 3 & 4 \\
5 & 6 & 7 & 8 \\
9 & 10 & 11 & 12 \\
13 & 14 & 15 & 16 \\
\end{array}
\]