PROBLEM SET 4: SOME RANDOM PROBLEMS

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Hand in four problems on Feb. 6.

1. IN-CLASS DISCUSSION PROBLEMS

Example 1. Suppose that a triangle is given. Describe how one can construct a square such that all 4 vertices of the square lie on the sides of the triangle.

Example 2. There are 25 matches on the table. Two players take turns. Each turn they have to take away 1, 2 or 3 matches. The person taking the last match loses. Show that the second player always can win this game.


Example 3. (Hungarian Eötvös competition) Find the sum of all four digit numbers which contain that contain only the digits 1, 2, 3, 4, 5, each at most once.

2. PROBLEMS

Problem 1. ** A man has 10 pockets and 44 silver dollars. He wants to put his dollars into his pocket so distributed that each pocket contains a different number of dollars.

(a). Can he do so?
(b). Generalize the problem considering p pockets and n dollars. The problem is most interesting when

\[ n = \frac{(p + 1)(p - 2)}{2}. \]

Why?
Problem 2. ** A bacterium splits into two identical ones with probability $p$, otherwise it dies. What is the probability that, beginning with this one bacterium, the colony lasts forever?

Problem 3. ** What is the maximum and the minimum value of

$$\frac{ab + bc + ca}{a^2 + b^2 + c^2}$$

where $a, b, c$ are real numbers, not all equal to 1. For which $a, b, c$ do we get this maximum or minimum value?

Problem 4. ** Suppose we have a scale (one with two arms which can only decide if two weights are equal or which one is heavier). There are 9 coins, of which one is counterfeit. All coins weight the same, except the counterfeit one, which is heavier. Show that weighing only twice, one can determine which coin is counterfeit.

Problem 5. *** Prove that every polygon of perimeter $2a$ can be completely covered by a circular disc of diameter $a$.

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Problem 7. *** Let $P(x)$ the the unique polynomial of degree $n$ such that

$$P(x) = \frac{x}{x + 1}, \quad \text{for } x = 0, 1, 2, \ldots, n$$

Determine $P(n + 1)$.

Problem 8. *** Five points lie in the unit square. Show that the distance between two of them is at most $\frac{1}{2}\sqrt{2}$.

Problem 9. **** If $a_0, a_1, \ldots, a_n$ are real numbers satisfying

$$\frac{a_0}{1} + \frac{a_1}{2} + \cdots + \frac{a_n}{n + 1} = 0$$

Show that the polynomial $a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$ has at least one real root.

Problem 10. ***** Evaluate

$$\sum_{n=2}^{\infty} \frac{\varphi(n)}{2^n - 1}$$

where $\varphi(n)$ is Euler’s function, so $\varphi(n)$ is the number of integers $m$ with $1 \leq m \leq n$ and $m$ and $n$ are relatively prime.