Problem 1. Prove the formula \(((A \& \sim B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \lor C))\).

1. Suppose that \((A \& \sim B) \Rightarrow C\)
2. Suppose \(A\)
3. \(B \lor \sim B\), so there are two cases:
4. **case 1:** Suppose \(B\)
5. \(B \lor C\)
6. **case 2:** Suppose \(\sim B\)
7. Combining (2) and (6) gives us \(A \& \sim B\)
8. By (1) and (7) we get \(C\)
9. \(B \lor C\)
10. In each of the two cases we have \(B \lor C\)
11. Because of the assumption in (2), we have \(A \Rightarrow B \lor C\)
12. Because of the assumption in (1), we conclude \(((A \& \sim B) \Rightarrow C) \Rightarrow (A \Rightarrow (B \lor C))\)

For problems 2, 3 and 4 let \(S\) be the statement:
For all points \(A, B, C\) which are not collinear, there exists a line which is not incident to \(A, B\) or \(C\).

**Problem 2.** Give the statement you get if you dualize the statement \(S\).

For all lines \(a, b, c\) which are not concurrent, there exists a point which is not incident to any of the lines \(a, b, c\).

**Problem 3.** Give a model of an affine plane for which statement \(S\) is false.

Take the model with points \(A, B, C, D\). The lines are all subsets of \(\{A, B, C, D\}\) with exactly 2 elements: \(\{A, B\}, \{A, C\}, \{A, D\}, \{B, C\}, \{B, D\}, \{C, D\}\). This is an affine plane (this example is given in the book). Now every line contains at least one of the points \(A, B, C\). So statement \(S\) is not true.

**Problem 4.** Show that statement \(S\) is true for any projective plane. (State which axioms you are using.)

Suppose that \(A, B, C\) are points which are not collinear. The line \(\overrightarrow{AB}\) (line \(\overrightarrow{AB}\) exists and is unique by Axiom I-1) has at least 3 points (by the Elliptic parallel axiom). So let \(D\) be a point on \(\overrightarrow{AB}\) which is not equal to \(A\) or \(B\). Similarly, let \(E\) be a point on \(\overrightarrow{AC}\) which is not equal to \(A\) or \(C\). Let \(\ell\) be the line \(\overrightarrow{DE}\). Suppose that \(\overrightarrow{DE}\) contains
A or B. Then $\overrightarrow{DE}$ and $\overrightarrow{AB}$ have at least 2 points in common, so $\overrightarrow{DE} = \overrightarrow{AB}$ by the uniqueness in axiom I-1. Since $\overrightarrow{AB}$ and $\overrightarrow{AC}$ both contain A and D, we have $\overrightarrow{AB} = \overrightarrow{AC}$. But then A, B, C are collinear. Contradiction. So $\ell$ does not contain A or B. A similar argument (after exchanging the roles of B and C) shows that $\ell$ cannot contain C.

**Problem 5.** Use the incidence axioms, and the betweenness axioms to prove the following: Suppose that A, B, C, D are 4 distinct points, no three on a line. If $\overrightarrow{AB}$ intersects $\overrightarrow{CD}$, then $\overrightarrow{AC}$ does not intersect $\overrightarrow{BD}$.

Suppose that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ intersect. Let Q be the intersection point. Then Q is not equal to A, B, C or D. We have $\overrightarrow{A\star Q\star C}$ and $\overrightarrow{B\star Q\star D}$. Since A, C, D are not collinear, we have that $\overrightarrow{AQ}$ is not equal to $\overrightarrow{CD}$. So C is the only point that $\overrightarrow{AQ}$ and $\overrightarrow{CD}$ have in common. If line segment AQ intersects with line CD, then the intersection point would have to be C. But C does not lie on the line segment AQ because $\overrightarrow{A\star Q\star C}$. Contradiction. So AQ does not intersect $\overrightarrow{CD}$. This shows that A and Q lie on the same side of $\ell$. A similar argument (by exchanging the roles of A and B and C and D) shows that B and Q lie on the same side of $\ell$. By axiom B-4, A and B lie on the same side of $\ell$. Hence, $\overrightarrow{AB}$ does not intersect $\overrightarrow{CD}$. But it is given that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ do intersect. Contradiction. We conclude that $\overrightarrow{AC}$ and $\overrightarrow{BD}$ do not intersect.