Problem 1. For this problem we assume we are dealing with a Hilbert plane.

(a) Suppose that $\ell$ and $m$ are two parallel lines, and $P$ is a point, not on $\ell$ or $m$. Suppose that there exists a point $Q$ on $\ell$ such that $P$ and $Q$ lie on the same side of $m$. Show that $P$ and $R$ lie on the same side of $m$ for every point $R$ on $\ell$. In this case we say that $P$ and $\ell$ lie on the same side of $m$.

The lines $\ell$ and $m$ are parallel, so they do not intersect. Therefore, $QR$ and $m$ do not intersect. By definition, $Q$ and $R$ lie on the same side of $m$. Since $P, Q$ on the same side, and $Q, R$ on the same side, we get $P$ and $R$ on the same side of $m$ by Axiom B-4.

(b) In the Poincaré upper half-plane model, give an example of three pairwise parallel lines such that none of the lines lies in between the other two. (A sketch suffices.) Explain why there is no line that intersects with all three lines.

There are many possible “sketches”. One could take for example the ray defined by $x = -2$ and $y \geq 0$ (call this line $\ell_1$), the ray defined by $x = 2$ and $y \geq 0$ (call this line $\ell_2$) and the semicircle defined by $x^2 + y^2 = 1, y \geq 0$ (call this $\ell_3$). These are three lines in the Poincaré plane. It is easy to see that there is a line $m$ that intersects only $\ell_1$ and $\ell_2$, in $P_1$ and $P_2$, say. If $\ell_3$ is in between $\ell_1$ and $\ell_2$ then $P_1P_2$ intersects $\ell_3$, so $m$ intersects $\ell_3$ as well. Contradiction. So $\ell_3$ is not in between $\ell_1$ and $\ell_2$. We can also find lines that only intersect $\ell_2$ and $\ell_3$, or only $\ell_1$ and $\ell_3$. So no line lines in between the other two.

(c) Suppose that $\ell, m, n$ are pairwise parallel lines. Show that if $m$ lies in between $\ell$ and $n$, then $\ell$ does not lie in between $m$ and $n$.

Suppose that $m$ lies in between $\ell$ and $n$. If we choose a point $P$ on $\ell$ and a point $Q$ on $n$, then $PQ$ intersects $m$ because $P$ and $Q$ are on opposite sides of $\ell$. Let $R$ be this intersection point. Now $QR$ does contain $P$, so $QR$ does not intersect $\ell$, and $Q$ and $R$ are not on opposite sides of $\ell$. Therefore, $\ell$ does not lie in between $m$ and $n$. 
Problem 2. Let $\mathbb{F}_5 = \{0, 1, 2, 3, 4\}$ be the field with 5 elements.

(a) How many points does the projective plane $\mathbb{P}^2(\mathbb{F}_5)$ have? How many lines? How many points does each line have? How many lines go through each point?

$\mathbb{P}^2(\mathbb{F}_q)$ has $q^2 + q + 1$ points, $q^2 + q + 1$ lines. Every line has $q + 1$ points and $q + 1$ lines go through every point. So $\mathbb{P}^2(\mathbb{F}_5)$ has $5^2 + 5 + 1 = 31$ points, 31 lines. Every line has $q + 1 = 6$ points and 6 lines go through every point.

(b) Give 5 points in $\mathbb{P}^2(\mathbb{F}_5)$ such that no three of them lie on a line.

$[1 : 0 : 0], [0 : 1 : 0], [0 : 0 : 1], [1 : 1 : 1], [1 : 2 : 3]$.

Problem 3. Assume the axioms of neutral geometry. Let $\triangle ABC$ be a triangle. Suppose that $P, Q, R$ are points on the line segments $AB, BC, AC$ respectively. Assume that $A, B, C, P, Q, R$ are 6 distinct points. Show that

\[
(\angle APQ)^\circ + (\angle PQR)^\circ + (\angle QRA)^\circ + (\angle RAP)^\circ > 180^\circ
\]

\[
(\angle APQ)^\circ > (\angle PQB)^\circ, (\angle QRA)^\circ > (\angle RQC)^\circ
\]

by the exterior angle theorem.

Problem 4. For a Hilbert plane, prove the following: If $\triangle ABC$ is a triangle with $\angle A > \angle B$ and $P$ is a point with $A \star P \star B$, then $PC < BC$.

$\angle BPC > \angle A > \angle B$ using the exterior angle theorem. Since $PC$ is opposite $\angle B$, and $BC$ is opposite $\angle BPC$ we have $PC < BC$.

Problem 5. In Neutral geometry, prove the following: Suppose that $ABDC$ is a bi-right quadrilateral where $\angle A, \angle B$ are right angles and the angles $\angle C, \angle D$ are acute. Prove that $\overrightarrow{AB}$ and $\overrightarrow{CD}$ are parallel.

Since $ABDC$ is a quadrilateral, $\overrightarrow{AB}$ and $CD$ do not intersect. Suppose $C \star D \star P$ and $P$ lies on $\overrightarrow{AB}$. Since $\angle PBD$ is right, $\angle PDB$ must be acute, but then $\angle BDC$ is obtuse. Contradiction. If $P \star C \star D$ we get a similar contradiction.

Problem 6. Give a model for incidence geometry with the following property: For every line $\ell$ and every point $P$ not on $\ell$, there are exactly 2 distinct lines through $P$ which are parallel to $\ell$.

Let $\{A, B, C, D, E\}$ be the points and all 10 subset with 2 elements the lines. If we take for example the line $\{A, B\}$ and the point $C$, then there exist two lines, namely $\{C, D\}$ and $\{C, E\}$ which are parallel to $\{A, B\}$ and which go through $C$. 
**Problem 7.** In neutral geometry, suppose that $\triangle ABC$ is a triangle and $M$ is the midpoint of $BC$. Show that $2 \cdot AM < AB + AC$.

Construct a point $A'$ such that $A, A'$ are on the opposite sides of $\overrightarrow{BC}$, and $\triangle ABC$ is congruent to $\triangle A'CB$. Let $N$ be the intersection point of $AA'$ and $BC$. $\angle ACB \cong \angle A'BC$, $\angle BCA' \cong \angle ABC$. By angle addition, $\angle ACA' \cong \angle BAA'$. By SAS, $\triangle ACA' \cong \triangle A'BA$, so $\angle CAA' \cong \angle BA'A$. By ASA, $\triangle A'NB \cong \triangle ANC$. So $NC \cong NB$ and $N = M$ is the midpoint. Now $AB + AC = AB + BA' > AA' = 2AM$. 