1. Let $x, y, x_1, x_2, x_3, x_4$ be indeterminates over the field $K$. The kernel $P$ of the $K$-algebra map $S = K[x_1, x_2, x_3, x_4] 	o K[x^2, x^3, xy, y] \subseteq K[x, y]$ such that $x_1, x_2, x_3, x_4$ are sent to $x^2, x^3, xy, y$, respectively, is the ideal generated by the $2 \times 2$ minors of the matrix

$$
\begin{pmatrix}
x_1 & x_2 & x_3 & x_4 \\
x_2 & x_1^2 & x_1 x_4 & x_3
\end{pmatrix},
$$

which is $(X_1^3 - X_2^2, X_1^2 X_4 - X_2 X_3, X_1 X_3 - X_2 X_4, X_3^2 - X_1 X_4^2)$. You may assume this. Let $X = V(P)$ and $Y = V(Q)$, where $Q = (x_2, x_4)$. Show that the intersection $X \cap Y$, as a set, consists only of one point, the origin, corresponding to $m = (x_1, x_2, x_3, x_4)$, and find the intersection multiplicity of $X$ and $Y$ at this point. I.e., with $T = S_m, \chi^T(T/PT, T/QT)$. What is $\ell(T/PT) \otimes T(QT))$?

2. Let $R$ be a regular local ring whose completion is formal power series over a field or a discrete valuation ring. Let $P, Q$ be prime ideals such that $\dim (R/P) + \dim (R/Q) = \dim (R)$ and $P + Q$ is $m$-primary. Let $M, N$ be nonzero torsion free modules over $R/P$ and $R/Q$, respectively, and suppose that the torsion-free rank of $M$ (resp., $N$) over $R/P$ (resp., over $R/Q$) is $r$ (resp., $s$). Show that $\chi^R(M, N) = rs \chi^R(R/P, R/Q)$.

3. Let $R$ be a regular local ring and let $M, N$ be nonzero finitely generated modules such that $\ell(M \otimes_R N)$ is finite. Suppose that $\text{depth}_m M = r = \dim (M)$, that $\text{depth}_m N = s = \dim (N)$ (i.e., $M$ and $N$ are Cohen-Macaulay modules), and that $r + s = d = \dim (R)$. Show that $\text{Tor}_i^R(M, N) = 0$ for $i \geq 1$. (Hence, $\chi(M, N) = \ell(M \otimes_R N) > 0$.)

4. Let $(R, m) \hookrightarrow S$ be flat local homomorphism of regular rings, let $\bar{x} = x_1, \ldots, x_d$ be a system of parameters for $R$, and let $x_1, \ldots, x_d, y_1, \ldots, y_r$ be a system of parameters for $S$. Let $\bar{y} = y_1, \ldots, y_r$, and let $\overline{S} = S/(\bar{y}) S$. Show that $\overline{S}$ is flat over $R$.

5. With $(R, m), S, T, \bar{x}$, and $\bar{y}$ as in problem 4., let $M, N$ be nonzero $R$-modules such that $\ell(M \otimes_R N)$ is finite. Let $B = S \otimes_R M$ and $C = \overline{S} \otimes_R N$. Show that $\chi^S(B, C)$ is defined, and that it is equal to $\ell_S(S/(\bar{y})) \chi^R(M, N)$.

6. Let $R \to S$ be a homomorphism of Noetherian rings such that $S$ has projective dimension at most $d$ as an $R$-module. Let $M$ be an $R$-module and let $N$ be an $S$-module of projective dimension at most $h$ over $S$. Show that $\text{Tor}_n^R(M, N)$ vanishes for $n > d + h$, and is the same as $\text{Tor}_h^S(\text{Tor}_d^R(M, S), N)$ for $n = d + h$.

Extra Credit 7. Assume that the normalization of a complete local two-dimensional domain $D$ is a module-finite extension of $D$. (It will then be a finitely generated $D$-module of depth two on the maximal ideal of $D$.) Prove that if $M, N$ are nonzero modules over an arbitrary four-dimensional regular local ring $R$ such that $\ell(M \otimes_R N) \dim (M) + \dim (N) = 4$, then $\chi^R(M, N) > 0$. As observed in class, you may assume that $M, N$ are prime cyclic modules.

Extra Credit 8. Let notation by as in Problem 1. Determine $\chi^T(T/P^rT, T/Q^sT)$ as a function of the positive integers $r$ and $s$. 

Math 615, Winter 2014
Problem Set #4
Due: Monday, April 6