0. Introduction

Throughout this paper $R$ is a Noetherian ring of prime characteristic $p$. By $(R, \mathfrak{m}, k)$, we indicate that $R$ is a local ring with maximal ideal $\mathfrak{m}$ and residue field $R/\mathfrak{m} = k$.

Also, we always use $q = p^e, Q = p^F, q_0 = p^{e_0}, q' = p^{e'}, q'' = p^{e''}$, etcetera, to denote varying powers of $p$ with $e, E, e_0, e', e'' \in \mathbb{N}$.

Let $M$ be an $R$-module. Then for any $e \geq 0$, we can derive a left $R$-module structure on the set $M$ by $r \cdot m := r^{p^e}m$ for any $r \in R$ and $m \in M$. For technical reasons, we keep the original right $R$-module structure on $M$ by default. We denote the derived $R$-$R$-bimodule by $^eM$. Thus, in $^eM$, we have $r \cdot m = m \cdot r^{p^e}$, which is equal to $r^{q}m$ in the original $M$. If $R$ is reduced, then $^eR$, as a left $R$-module, is isomorphic to $R^{1/q} := \{r^{1/p} \mid r \in R\}$. We use $\lambda^e(-), \lambda^e(-)$ to denote the left and right lengths of a bimodule. It is easy to see that $\lambda^e(\mathcal{M}) = q^{\alpha(R)}\lambda^e(\mathcal{M}) = q^{\alpha(R)}\lambda(M)$ for any finite length module $M$ over $(R, \mathfrak{m}, k)$, in which $\alpha(R) = \log_p[k : k^p]$.

We say that $R$ is $F$-finite if $^1R$ (or, equivalently, $^eR$ for all $e$) is finitely generated as an left $R$-module.

For any $R$-module $M$ and $e$, we can always form a new $R$-module $F^e(M)$ by scalar extension via $F^e : R \rightarrow R$ by $r \mapsto r^{q}$. In other words, $F^e(M)$ has the $R$-module structure that is determined by the right $R$-module structure of $M \otimes_R \mathcal{M}$; and it is this $R$-module structure of $F^e(M)$ that we mean unless otherwise specified. If $h \in \text{Hom}_R(M, N)$, then we correspondingly have $F^e(h) : \text{Hom}_R(F^e(M), F^e(N))$. Sometimes, especially when both $M$ and $N$ are free, we may write $F^e(h)$ as $h^{[\alpha]}$.

A very important concept in studying rings of characteristic $p$ is tight closure. Tight closure was first studied and developed by Hochster and Huneke in the 1980’s.
**Definition 0.1** (Hochster-Huneke, [HH1]). Let $R$ be a Noetherian ring of prime characteristic $p$ and $N \subseteq M$ be $R$-modules. The tight closure of $N$ in $M$, denoted by $N^*_M$, is defined as follows: An element $x \in M$ is said to be in $N^*_M$ if there exists an element $c \in R^\circ$ such that $x \otimes c \in N_{M}^{[q]} \subseteq M \otimes_R eR$ for all $e \gg 0$, where $R^\circ$ is the complement of the union of all minimal primes of the ring $R$ and $N_{M}^{[q]}$ denotes the (right) $R$-submodule of $F_{R}^c(M) = M \otimes_R eR$ generated by $\{x \otimes 1 \in M \otimes_R eR \mid x \in N\}$. The element $x \otimes 1 \in M \otimes_R eR$ is denoted by $x^q_M = x^q_M$. (By our convention on $F_{R}^c(M)$, we have $cx^q_M = x \otimes c \in N_{M}^{[q]}$.)

**Definition 0.2** ([HH2]). Let $R$ be a Noetherian ring of prime characteristic $p$, $q_0 = p^{e_0}$ and let $N \subseteq M$ be $R$-modules. We say $c \in R^\circ$ is a $q_0$-weak test element for $N \subseteq M$ if $c(N_M^* \setminus [q]) \subseteq N_{M}^{[q]}$ for all $q \geq q_0$. In case $N = 0$, we may simply call it a test element for $M$. By a $q_0$-weak test element, we simply mean a $q_0$-weak test element for all $R$-modules. If a $q_0$-weak test element $c$ remains a $q_0$-weak test element under every localization, then we call $c$ a locally stable $q_0$-weak test. Finally, in case $q_0 = 1$, we simply call $c$ a test element or locally stable test element.

**Definition 0.3** ([HH3]). Let $R$ be a Noetherian ring of prime characteristic $p$, $c \in R$, and $N \subseteq M$ (finitely generated) $R$-modules. We say that $Q = p^E$ is a test exponent for $c$ and $N \subseteq M$ (over $R$) if, for any $x \in M$, the occurrence of $cx^q \in N_{M}^{[q]}$ for one single $q \geq Q$ implies $x \in N_M^*$. In case $N = 0$, we may simply call it a test exponent for $c$ and $M$.

**Remark 0.4.** (1) It is easy to check the following statements: To say $c \in R^\circ$ is a test element for $N \subseteq M$ is the same as to say $c$ is a test element for $(0 \subseteq M/N)$ $M/N$. Similarly, to say $Q = p^E$ is a test exponent for $c$ and $N \subseteq M$ is the same as to say $Q$ is a test exponent for $c$ and $(0 \subseteq M/N)$ $M/N$.

(2) However, by ‘a ($q_0$-weak) test element for an ideal $I$’, we usually mean ‘a ($q_0$-weak) test element for $I \subseteq R$’ rather than ‘a ($q_0$-weak) test element for $0 \subseteq I$’. Similarly, when we say ‘a test exponent for $c$ and an ideal $I$’, we usually mean ‘a test exponent for $c$ and $I \subseteq R$’ rather than ‘a test exponent for $c$ and $0 \subseteq I$’.

Under mild conditions, test elements exist.

**Theorem 0.5.** Let $R$ be $F$-finite or essentially of finite type over an excellent local ring $(A, \mathfrak{n})$ of characteristic $p$. Say $\sqrt{0^{[q_0]}} = 0$, where $\sqrt{0}$ is the nilradical of $R$.

(1) There exists a completely stable $q_0$-weak test element for all finitely generated $R$-modules. (See [HH2].)

(2) In fact, there exists a completely stable $q_0$-weak test element for all (not necessarily finitely generated) $R$-modules. (It suffices to prove the case where $R$ (and hence $A$) is reduced. Under the assumption that $R$ is $F$-finite, this was proved in the thesis of Haggai Elitzur, [El]. From this we can see the remaining case via a faithfully flat extension $R \rightarrow R \otimes_A \hat{A} \rightarrow R \otimes_A \hat{A}^\Gamma$, where $\hat{A}$ is the $\mathfrak{n}$-adic completion of $A$ and $\hat{A}^\Gamma$ is a suitable ($F$-finite) $\Gamma$-extension of $\hat{A}$. See [HH2] for details about $\Gamma$-extensions.)
If there exists a test exponent for a locally stable test element \( c \in R^\circ \) and (finitely generated) \( R \)-modules \( N \subseteq M \), then the tight closure of \( N \) in \( M \) commutes with localization. This result is implicit in \([\text{McD}]\) and is explicitly stated in \([\text{HH3}]\) Proposition 2.3. Moreover, Hochster and Huneke showed in \([\text{HH3}]\) that the converse is true as below.

**Theorem 0.6 \([\text{HH3}]\).** Let \( R \) be a Noetherian ring of prime characteristic \( p \) with a given locally stable test element \( c \), and \( N \subseteq M \) finitely generated \( R \)-modules. Assume that the tight closure of \( N \) in \( M \) commutes with localization. Then there exists a test exponent for \( c \) and \( N \subseteq M \).

In \([\text{HH3}]\), Hochster and Huneke asked, among other questions, whether there exists a uniform test exponent for a given test element and all ideals generated by systems of parameters. This question has been recently answered positively by R. Y. Sharp.

**Theorem 0.7 \([\text{Sharp, Sh, Theorem 3.2}]\).** Let \( (R, \mathfrak{m}) \) be an equidimensional excellent local ring of prime characteristic \( p \) and \( c \in R^\circ \). Then there exists a test exponent for \( c \) and all ideals generated by (partial or full) systems of parameters of \( R \).

In Theorem 2.4 we use the Artinian property of \( H^{\dim(R)}(R) \) and colon-capturing to give an alternate proof of the above Theorem 0.7.

Inspired by Sharp’s result, we then naturally ask whether there is a uniform test exponent for a given \( c \in R^\circ \) and all finitely generated \( R \)-modules with (finite length and) finite phantom projective dimension. While this question remains unsettled, we can give an affirmative answer in case \( R \) is Cohen-Macaulay or in case \( \dim(R) \leq 2 \).

Throughout this paper, we use \( \lambda(M) \) and \( \text{ppd}(M) \) to denote the length and phantom projective dimension of an \( R \)-module \( M \) respectively.

**Theorem \((\text{Corollary 3.3, Corollary 3.4})\).** Let \( (R, \mathfrak{m}) \) be an equidimensional Noetherian excellent local ring of prime characteristic \( p \). Assume either that \( R \) is Cohen-Macaulay or \( \dim(R) \leq 2 \). Then, for any \( c \in R^\circ \), there is a test exponent for \( c \) and all \( R \)-modules \( M \) with \( \lambda(M) < \infty \), \( \text{ppd}(M) < \infty \).

For any finitely generated \( R \)-modules \( L, M \) such that \( \lambda(L) < \infty \), we denote

\[
e_{HK}(L, M) = \lim_{q \to \infty} \lambda(F^e(L) \otimes_R M)/q^\dim(R) = \lim_{q \to \infty} \lambda^r(L \otimes_R eM)/q^\dim(R),
\]

whose existence is ensured by a result of G. Seibert in \([\text{Se}]\). In case \( L = R/I \), we often denote \( e_{HK}(R/I, M) \) by \( e_{HK}(I, M) \) in spite of the remote possibility of confusion. It is easy to see that \( e_{HK}(I, M) = \lim_{q \to \infty} \lambda(M/I[q]M)/q^\dim(R) \) and the existence of \( e_{HK}(I, M) \) was first proved by P. Monsky in \([\text{Mo}]\).

Next we connect the above results about uniform test exponents for all modules of finite length and of finite phantom projective dimension to results about the \( F \)-rational signature of \( R \). Here we mention that \( r_R(M) \) has been defined and studied in \([\text{HY}]\). Throughout this paper, s.o.p. is short for “system of parameters.”

**Definition 0.8.** Let \( (R, \mathfrak{m}) \) be a local ring of prime characteristic \( p \) and \( M \) a finitely generated \( R \)-module.
(1) Define (see [HY])

\[ r_R(M) = \inf \{ e_{HK}(x, M) - e_{HK}(J, M) \mid x \text{ is a s.o.p. for } R \text{ and } (x) \subseteq J \}. \]

In particular, \( r_R(R) \) is called the \( F \)-rational signature of \( R \).

(2) In case \( (R, m) \) is such that \( \text{ppd}(R/(x)) < \infty \) for every system of parameters \( x \) of \( R \) (e.g., \( R \) is an equidimensional homomorphic image of a Cohen-Macaulay ring or \( R \) is an equidimensional excellent ring), we define

\[ r'_R(M) = \inf \{ e_{HK}(L, M) - e_{HK}(L/K, M) \mid \text{ppd}(L) < \infty, \lambda(L) < \infty, 0 \neq K \subseteq L \}. \]

Otherwise, we define \( r'_R(M) = 0 \). We call \( r'_R(R) \) the phantom \( F \)-rational signature of \( R \).

**Theorem** (Theorem 1.2). Let \( (R, m) \) be a Noetherian local ring of prime characteristic \( p \). Assume there exists a common weak parameter test element for \( R \) and \( \hat{R} \) (e.g., \( R \) is excellent). Then \( R \) is \( F \)-rational \( \iff \hat{R} \) is \( F \)-rational \( \iff r'_R(M) > 0 \iff r'_R(M) > 0 \) for every (or, equivalently, some) finitely generated \( R \)-module with \( \dim(M) = \dim(R) \).

### 1. Some preliminary results about test exponents

We first observe the following easy lemma about test exponents, although it is not directly used in the sequel.

**Lemma 1.1.** Let \( R \) be a Noetherian ring of characteristic \( p \). For any \( b, c \in R^\circ \) and \( R \)-modules \( N \subseteq M \), the following are true.

1. If \( Q \) is a test exponent for \( bc \) and \( N \subseteq M \), then \( Q \) is a test exponent for \( c \) and \( N \subseteq M \).

2. If, for some \( q_0 = p^{e_0} \), \( Q \) is a test exponent for \( c^{q_0} \) and \( N^{[q_0]}_M \subseteq F^{e_0}_R(M) \), then \( Q \) is a test exponent for \( c \) and \( N \subseteq M \).

**Proof.** 1. If \( cx^q \in N^{[q]}_M \subseteq F^{e}_R(M) \) for some \( x \in M \) and \( p^e = q \geq Q \), then \( bce^q \in N^{[q]}_M \subseteq F^{e}_R(M) \) and hence \( x \in N^{*_e}_M \).

2. Suppose \( cx^q \in N^{[q]}_M \subseteq F^{e}_R(M) \) for some \( x \in M \) and \( p^e = q \geq Q \). Then \( c^{q_0}x^{q_0q} \in N^{[q_0]}_M \subseteq F^{e_0+e}_R(M) \), or, in other words, \( c^{q_0}(x^{q_0})^q \in (N^{[q_0]}_M)^{[q]}_{F^{e_0}_R(M)} \subseteq F^{e}_R(F^{e_0}_R(M)) \). This implies \( x^{q_0} \in (N^{[q_0]}_M)^{e_0}_F(M) \), which forces \( x \in N^{*_e}_M \).

For simplicity, we state the next two results (i.e., Lemma 1.2 and Lemma 1.3) in terms of test exponent for \( c \) and \( (0 \subseteq M) \) only. It is an easy task to give the corresponding statements in terms of test exponents for \( c \) and \( N \subseteq M \).

**Lemma 1.2.** Let \( R \) be a Noetherian ring of characteristic \( p \) with the set of minimal primes \( \text{min}(R) = \{ P_1, P_2, \ldots, P_r \} \) so that \( \sqrt{0} = \cap_{i=1}^r P_i \). For any \( c \in R^\circ \) (or simply \( c \in R \)) and any (finitely generated) \( R \)-module \( M \), the following statements are true.

1. If \( Q \) is a test exponent for \( c + P_i \) and \( M/P_iM \) over \( R/P_i \) for all \( i = 1, 2, \ldots, r \), then \( Q \) is a test exponent for \( c + M \) and \( M \).

2. If \( Q \) is a test exponent for \( c + \sqrt{0} \) and \( M/\sqrt{0}M \) over \( R/\sqrt{0} \), then \( Q \) is a test exponent for \( c + M \).
Proof. (1). Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then, $(c+P_i)(x+P_i M)^e_{M/P_i M} = 0 \in F_R^e(M/P_i M)$, which implies $x+P_i M \in 0^e_{M/P_i M}$ for every $i = 1, 2, \ldots, r$. This forces $x \in 0^e_M$ (see [HH1]).

(2). This follows similarly. \hfill \Box

The next lemma deals with module-finite and pure ring extensions. In particular, the lemma applies to any reduced Nagata (e.g., excellent) ring and its integral closure in its total quotient ring.

**Lemma 1.3.** Let $R \subseteq S$ be an extension of Noetherian rings of characteristic $p$, $c \in R$, and let $M$ be a (finitely generated) $R$-module. Assume either (1) $R \subseteq S$ is module-finite, or (2) $R \subseteq S$ is a pure extension with a common weak test element in $R$. If $Q$ is a test exponent for $c$ and $0 \subseteq M \otimes_R S$ over $S$, then $Q$ is a test exponent for $c$ and $0 \subseteq M$.

Proof. Suppose $cx^q = 0 \in F_R^e(M)$ for some $x \in M$ and $p^e = q \geq Q$. Then $c(x \otimes 1)^q = 0 \in F_S^e(M \otimes_R S)$ and hence $x \otimes 1 \in 0^e_{M \otimes_R S}$, which implies $x \in 0^e_M$. \hfill \Box

The next lemma relies on the ‘colon-capturing’ property of tight closure, which is systematically studied in [HH1], Section 7.

**Lemma 1.4.** Let $(R, m)$ be a Noetherian local ring of characteristic $p$, $\dim(R) = d$, and $x = x_1, x_2, \ldots, x_d$ and $y = y_1, y_2, \ldots, y_d$ be two systems of parameters such that $(y) \subseteq (x)$. For each $j = 1, 2, \ldots, d$, say $y_j = \sum_{i=1}^{d} x_i a_{ij}$ with $a_{ij} \in R$. Denote the resulting $d \times d$ matrix $(a_{ij})_{d \times d}$ by $A$. Then

1. $(y)^*:R (x) \supseteq (\det(A))^*$ and $(y)^*:R \det(A) \supseteq (x)^*$.

Further assume that $(R, m)$ is equidimensional and, moreover, that $R$ is either excellent or a homomorphic image of a Cohen-Macaulay ring. Then

2. $(y)^*:R (x) = ((y) + (\det(A)))^*$ and $(y)^*:R \det(A) = (x)^*$.

3. For any $c \in R$, if $Q$ is a test exponent for $c$ and $(y) \subseteq R$, then $Q$ is a test exponent for $c$ and $(x) \subseteq R$.

Proof. (1). This is straightforward (cf. [HH1] Proposition 4.1(b)(k)).

To prove (2) and (3), we may assume $(R, m)$ is an equidimensional homomorphic image of a Cohen-Macaulay ring without loss of generality. (Indeed, in case $R$ is equidimensional and excellent, it suffices to prove (2) and (3) for $\widehat{R}$.)

(2). This can be proved similarly as [HH1] Theorem 7.9. In case there is no explicit proof available in the literature, we mention that this can be proved, as usual, by lifting everything up to the Cohen-Macaulay ring of which $R$ is a homomorphic image.

(3). Suppose $cx^q \in (x)^q$ for some $x \in R$ and $q \geq Q$. Then $c(\det(A)x)^q = \det(A)^q cx^q \in (y)^q$ and hence $\det(A)x \in (y)^*$, which implies $x \in (y)^*:R \det(A) = (x)^*$ by part (2) above. \hfill \Box

2. Test exponents for Artinian modules and
AN ALTERNATIVE PROOF OF SHARP’S THEOREM

We first show result about the existence of a test exponent for Artinian modules. Although the argument can be traced back to [HH3] (for modules of finite length), we include a proof here for the sake of convenience and completeness.

**Proposition 2.1** (Compare with [HH3, Proposition 2.6]). Let \( R \) be a Noetherian ring of prime characteristic \( p \) and \( N \subseteq M \) be \( R \)-modules such that \( M/N \) is Artinian. Assume there exists \( d \in \mathbb{N} \) that is a \( q_0 \)-weak test element for \( N^q_M \subseteq F^q_R(M) \) for all \( q \gg 0 \). Then, for any \( c \in R^\circ \), there exists a test exponent for \( c \) and \( N \subseteq M \).

**Proof.** For every \( e \), let \( N_e = \{ u \in M \mid cu^q \in (N^q_M)^F_{F^q(M)} \} \). Then, as shown in the proof of [HH3, Proposition 2.6], \( N_1 \supseteq N_2 \supseteq \cdots \supseteq N_e \supseteq N_{e+1} \supseteq \cdots \supseteq N \) and hence there exists \( Q = p^E \) such that \( N_e = N_E \) for all \( e \geq E \).

Suppose \( cx^c_q \in N^q_M \) for some \( x \in M \) and \( q' \geq Q \). Then \( x \in N_{q'} \) and thus \( x \in N_e \) for all \( e \geq E \). This means \( cx^c_q \in (N^q_M)^F_{F^q(M)} \subseteq (N^q_M)^{q'}_{F^{q'}(M)} \) for all \( q \geq Q \). Consequently, \( dx^{q_0}x^{q_0} = d(cx^c)^{q_0} = (N^q_M)^{q_0}_{F^{q_0}(M)} \) for all \( q \gg Q \), which implies \( x \in N_M^* \).

In light of Theorem [0.3], we get the following consequence of Proposition 2.1.

**Theorem 2.2.** Let \( R \) be an algebra essentially of finite type over an excellent local ring of characteristic \( p \), \( c \in R^\circ \), and \( M \) an Artinian \( R \)-module. Then there exists a test exponent for \( c \) and \( M \).

**Proof.** This follows immediately from Theorem 0.5(2) and Proposition 2.1.

We may refine Proposition 2.1 as follows when the Artinian \( R \)-module is the highest local cohomology.

**Proposition 2.3.** Let \((R, \mathfrak{m})\) be a Noetherian local ring of prime characteristic \( p \) and \( c \in R^\circ \). Assume \((R, \mathfrak{m})\) has the colon-capturing property and there exists a \( q_0 \)-weak test element \( b \in R^\circ \) for all parameter ideals of \( R \). Then there exists a test exponent for \( c \) and \( 0 \subset H^d_{\mathfrak{m}}(R) \).

**Proof.** Say \( \dim(R) = d \). Then \( H^d_{\mathfrak{m}}(R) = \lim_{\rightarrow} F^e_{(R)(R)} \). For any \( u \in R \) and any system of parameters \( \underline{x} = x_1, \ldots, x_d \) of \( R \), denote the image of \( u \) in \( H^d_{\mathfrak{m}}(R) \) by \( [\frac{u}{(x_1, \ldots, x_d)}] \). Recall that, for any \( e \in \mathbb{N} \), there is a canonical isomorphism \( F^e_{R}(H^d_{\mathfrak{m}}(R)) \cong H^d_{\mathfrak{m}}(R) \), under which we may simply write \( [\frac{u}{(x_1, \ldots, x_d)}]_{H^d_{\mathfrak{m}}(R)} = [\frac{u^q}{(x_1, \ldots, x_d)}] \). By colon-capturing, we see that \( [\frac{u}{(x_1, \ldots, x_d)}] \notin 0^e_{H^d_{\mathfrak{m}}(R)} \) if and only if \( u \in (x_1, \ldots, x_d)_{R}^* \) (cf. [Sm, Proposition 2.5]). This implies that \( b \) is a weak test element for \( 0 \subset H^d_{\mathfrak{m}}(R) \). (Indeed, for any \( \{ \frac{u}{(x_1, \ldots, x_d)} \} \notin 0^e_{H^d_{\mathfrak{m}}(R)} \), we have \( u \in (x_1, \ldots, x_d)_{R}^* \). Then \( bu^q \in (x_1, \ldots, x_d)_{R}^* \) for all \( q \geq q_0 \), which implies \( b[\frac{u}{(x_1, \ldots, x_d)}]_{H^d_{\mathfrak{m}}(R)} = [\frac{bu^q}{(x_1, \ldots, x_d)}] = 0 \in F^e_{R}(H^d_{\mathfrak{m}}(R)) \) for all \( q \geq q_0 \).

Consequently, \( b \) is a weak test element for \( 0 \subset F^e_{R}(H^d_{\mathfrak{m}}(R)) \) for all \( e \in \mathbb{N} \). Thus, by Proposition 2.1, there exists a test exponent, say \( Q = p^E \), for \( c \) and \( H^d_{\mathfrak{m}}(R) \).

Now we are ready to give a new proof of Sharp’s result about a uniform test exponent for \( c \in R^\circ \) and all ideals generated by systems of parameters.
Theorem 2.4 (R. Y. Sharp, [Sh Theorem 3.2]). Let \((R, m)\) be an equidimensional excellent local ring of prime characteristic \(p\) and \(c \in R^0\). Then there exists a test exponent for \(c\) and all ideals generated by (partial or full) systems of parameters of \(R\).

Proof. Say \(\dim(R) = d\). By Proposition 2.3 there is a test exponent \(Q\) for \(c\) and \(H^n_m(R)\). Here we keep the same usage of \(\lfloor \frac{u}{(x_1, \ldots, x_d)} \rfloor\) as in the above proof of Proposition 2.3.

Now, it suffices to show that \(Q\) is a test exponent for \(c\) and \((x_1, \ldots, x_i) \subseteq R\) for any (partial or full) system of parameters \(\underline{x} = x_1, \ldots, x_i\) of \(R\). But, then, it suffices to verify the case where \(\underline{x} = x_1, \ldots, x_d\) is any full system of parameters of \(R\) via a standard technique (see the last paragraph of the proof of [Sh Theorem 3.2]).

Finally, for any \(u \in R\) and \(q \geq Q\), suppose \(cu^q \in \langle x \rangle^{[q]} = (x_1^q, \ldots, x_d^q)\). This implies \(c[\lfloor \frac{u}{(x_1, \ldots, x_d)} \rfloor]^q_{H^n_m(R)} = 0 \in F^e_R(H^d_m(R))\). Thus, by the choice of \(Q\), \(\lfloor \frac{u}{(x_1, \ldots, x_d)} \rfloor \in 0^*_{H^n_m(R)}\), which forces \(u \in (x_1, \ldots, x_d)^*_R\) by colon-capturing as in Proposition 2.3 (cf. [Sm, Proposition 2.5]). \(\square\)

Next, we state a corollary of the theorem above.

Corollary 2.5. Let \((R, m)\) be an equidimensional excellent local ring of prime characteristic \(p\) and \(c \in R^0\). Then there exists a test exponent for \(c/1\) and all ideals generated by (partial or full) systems of parameters of \(R_P\) (over \(R_P\)) for all \(P \in \text{Spec}(R)\).

Proof. By Theorem 2.4 there is a test exponent, \(Q = p^F\), for \(c\) and all ideals generated by (partial or full) systems of parameters of \(R\). Fix an arbitrary \(P \in \text{Spec}(R)\). It suffices to show that \(Q\) is a test exponent for \(c/1\) and all ideals generated by (partial or full) systems of parameters of \(R_P\) (over \(R_P\)). Then, again, it suffices show that \(Q\) is a test exponent for \(c\) and all ideals generated by (full) systems of parameters of \(R_P\) (over \(R_P\)).

Say \(\dim(R_P) = h\). Then by prime avoidance, there exists \(\underline{x} = x_1, \ldots, x_h \in P\) such that \(\underline{x}\) is a (partial) system of parameters of \(R\). Then, for any \(0 < n \in \mathbb{N}\), \(\underline{x}^n := x^n_1, \ldots, x^n_h\) is also a (partial) system of parameters of \(R\) and, moreover, \(x^n_1/1, \ldots, x^n_h/1\) is a (full) system of parameters of \(R_P\).

Let \(y = y_1, \ldots, y_h\) be any full system of parameters of \(R_P\). We need to prove that \(Q\) is a test exponent for \(c/1\) and \((y) \subseteq \langle y \rangle_{R_P}\) in order to finish the proof. As there exists a positive integer \(n \in \mathbb{N}\) such that \((x^n_1, \ldots, x^n_h)^P \subseteq \langle y \rangle\), it suffices to prove that \(Q\) is a test exponent for \(c/1\) and \((x^n_1, \ldots, x^n_h)^P \subseteq \langle y \rangle_{R_P}\) by Lemma 1.4(3).

Now suppose \((c/1)v^q \in (x^n_1, \ldots, x^n_h)^{[q]}_{R_P}\) for some \(v \in R_P\) and \(q \geq Q\). Without loss of generality, we may assume \(v = u/1\) with \(u \in R\). That is, there exists \(s \in R\) such that \(scu^q \in (x^n_1, \ldots, x^n_h)^{[q]}\). Hence \(c(su)^q \in (x^n_1, \ldots, x^n_h)^{[q]}_{R}\), which implies \(su \in (x^n_1, \ldots, x^n_h)^*_R\). Therefore, \(v = u/1 \in (x^n_1, \ldots, x^n_h)^*_R \subseteq \langle (x^n_1, \ldots, x^n_h)^P \rangle_{R_P}^*\). \(\square\)

3. Modules with finite (phantom) projective dimension

Question 3.1. Assume \((R, m)\) is an equidimensional local ring of prime characteristic \(p\) that is either excellent or a homomorphic image of a Cohen-Macaulay ring. For
a given \( c \in R^c \), does there exist a test exponent for \( c \) and all finitely generated \( R \)-modules of finite phantom projective dimension?

If \( R \) is Cohen-Macaulay, then it is known that phantom projective dimension is the same as projective dimension. For this reason, the following theorem may be viewed as a partial answer to the above question.

**Theorem 3.2.** Let \( (R, \mathfrak{m}) \) be a Cohen-Macaulay Noetherian local ring of prime characteristic \( p \) with \( \dim(R) = d \). Fix any \( c \in R \), if \( Q = p^E \) is a test exponent for \( c \) and all ideals generated by (full) systems of parameters of \( R \), then \( Q \) is a test exponent for \( c \) and all \( R \)-modules of finite length and of finite projective dimension.

**Proof.** Let \( M \neq 0 \) be a typical \( R \)-module such that \( \lambda(M) < \infty \) and pd(\( M \)) < \( \infty \). Suppose \( cu^d = 0 \in F^e(M) \) for some \( u \in M, q' \geq Q \). We need to show \( u \in 0_M \).

Fix a minimal projective resolution \( G_\bullet \) of \( M \) as follows

\[
G_\bullet : \quad 0 \rightarrow G_d \xrightarrow{\phi_d} G_{d-1} \xrightarrow{\phi_{d-1}} \cdots \xrightarrow{\phi_1} G_1 \xrightarrow{\phi_0} G_0 \rightarrow 0.
\]

Then choose a system of parameters \( \underline{x} \) of \( R \) such that \( (\underline{x}) \subseteq \text{Ann}_R(u) \) and construct the Koszul complex \( K_\bullet(\underline{x}, R) \) as follows

\[
K_\bullet(\underline{x}, R) : \quad 0 \rightarrow K_d \xrightarrow{\psi_d} K_{d-1} \xrightarrow{\psi_{d-1}} \cdots \xrightarrow{\psi_1} K_1 \xrightarrow{\psi_0} K_0 \rightarrow 0,
\]

where \( K_i = R^{(d)} \). In particular, \( \psi_d \) is represented by matrix \( (x_1, x_2, \ldots, x_d) \) and the 0-th homology of \( K_i(\underline{x}, R) \) is \( R/(\underline{x}) \). Thus the \( R \)-linear map \( h : R/(\underline{x}) \rightarrow M = H_0(G_\bullet) \) sending the class of 1 to \( u \) can be lifted to a chain map \( g : K_\bullet(\underline{x}, R) \rightarrow G_\bullet \).

Denote \( g_0(1) = y \). Then \( cy^d \in (\text{Image}(\phi_1))^{[q]}_{G_0} \) and we now only need to show \( y \in (\text{Image}(\phi_1))^{[q]}_{G_0} \).

For every \( q \), there is an induced \( R \)-linear chain map \( g^{[q]} : F^e(K_\bullet(\underline{x}, R)) \rightarrow F^e(G_\bullet) \).

Now the fact that \( cy^d \in (\text{Image}(\phi_1))^{[q]}_{G_0} \) (i.e., \( cu^{q'} = 0 \)) implies that the chain map \( cg^{[q']}_{d-1} \) is homotopic to the zero chain map. In particular, there exists \( \delta_{d-1} \in \text{Hom}_R(F^e(K_{d-1}), F^e(G_d)) \) such that \( cg^{[q']}_{d-1} = \delta_{d-1} \circ \psi^{[q']}_d \). Applying \( \text{Hom}_R(-, R) \), we get

\[
c(\text{Image}(\text{Hom}(g_d, R)))^{[q]}_{K_d} = \text{Image}(\text{Hom}(cg^{[q']}_{d-1}, R))
\]

\[
= \text{Image}(\text{Hom}(\psi^{[q']}_{d-1}, R)) = (\underline{x})^{[q]}_R,
\]

which implies \( \text{Image}(\text{Hom}(g_d, R)) \subseteq (\underline{x})^{[q]}_R \) since \( q' \geq Q \). That is to say that there exists \( b \in R^c \) such that

\[
\text{Image}(\text{Hom}(bg^{[q]}_d, R)) = b \text{Image}(\text{Hom}(g^{[q]}_d, R))
\]

\[
= b(\text{Image}(\text{Hom}(g_d, R)))^{[q]}_R \subseteq (\underline{x})^{[q]}_R = \text{Image}(\text{Hom}(\psi^{[q]}_d, R))
\]

for all \( q \gg 0 \). Therefore, the chain maps

\[
\text{Hom}(bg^{[q]}_d, R) : \text{Hom}(F^e(G_\bullet), R) \rightarrow \text{Hom}(F^e(K_\bullet(\underline{x}, R)), R)
\]

are homotopic to 0 for all \( q \gg 0 \). Hence, there exist \( \epsilon^{[q]}_1 \in \text{Hom}_R(F^eG_1, F^e(K_0)) \) such that \( \text{Hom}(bg^{[q]}_d, R) = \epsilon^{[q]}_1 \circ \text{Hom}(\psi^{[q]}_d, R) \) for all \( q \gg 0 \). This, after going through
Hom(−, R), would in turn imply

\[ by^q \in b(\text{Image}(g_0))_{G_0}^{[q]} = \text{Image}(b|_{G_0}^{[q]}) \subseteq \text{Image}(\phi_1^{[q]}) = (\text{Image}(\phi_1))_{G_0}^{[q]}, \]

for all \( q \gg 0 \). We now conclude that \( y \in (\text{Image}(\phi_1))_{G_0}^{[q]} \) and the proof is complete. We also remark that the above argument of using homotopy to determine membership in the tight closure has appeared in [Ab].

**Corollary 3.3.** Let \((R, m)\) be a Cohen-Macaulay Noetherian excellent local ring of prime characteristic \( p \). Then, for any \( c \in R^\circ \), there is a test exponent for \( c \) and all \( R \)-modules of finite length and of finite (phantom) projective dimension.

**Proof.** This follows from Theorem 0.7 and Theorem 3.2.

We also notice that Question 3.1 reduces to the Cohen-Macaulay case if \( \dim(R) \leq 2 \).

**Corollary 3.4.** Let \((R, m)\) be an equidimensional excellent Noetherian local ring of prime characteristic \( p \) with \( \dim(R) \leq 2 \). Then, for any given \( c \in R^\circ \), there exists a test exponent for \( c \) and all \( R \)-modules of finite length and of finite phantom projective dimension.

**Proof.** By [HH1, Definition 9.1], we observe that any \( R \)-module of finite length and of finite phantom projective dimension over \( R \) remains so after we extend the scalar to the integral closure of \( R/P \) in its fraction field for every \( P \in \min(R) \). Therefore, by Lemma 1.2 and Lemma 1.3, we may assume \( R \) is normal without loss of generality. (We may assume \( R \) is complete as well.) But now \( R \) is excellent Cohen-Macaulay and the claim follows from Corollary 3.3.

4. A connection with \( F \)-rational signature

The \( F \)-rational signature, \( r_R(M) \), has been studied in [HY]. In this section, we investigate the behavior of \( r'_R(M) \) (cf. Definition 0.8(2)). We start with a remark.

**Remark 4.1.** First, from definition, it is immediate to see that \( r(M) \geq r'(M) \) for any finitely generated \( R \)-module \( M \).

Second, suppose that \( \text{ppd}(R/(x)) < \infty \) for every system of parameters \( x \) of \( R \). Then we observe that \( r'_R(M \otimes_R \hat{R}) \leq r'_R(M) \) for any finitely generated \( R \)-module \( M \). The reason is that any \( R \)-module of finite length and of finite phantom projective dimension remain so considered as an \( \hat{R} \)-module.

It turns out that \( r(M) \) and \( r'(M) \) behave quite similarly. For example, assuming \( R \) is excellent, the positiveness of \( r'(R) \) characterizes \( F \)-rationality.

**Theorem 4.2.** Let \((R, m)\) be a Noetherian local ring of prime characteristic \( p \). And we use \( M \) to denote a finitely generated \( R \)-module. Consider

(1) \( r(R) > 0 \); (1') \( r'(R) > 0 \);
(2) \( r(M) > 0 \) for every \( M \) with \( \dim(M) = \dim(R) \);
(2') \( r'(M) > 0 \) for every \( M \) with \( \dim(M) = \dim(R) \);
(3) \( r(M) > 0 \) for some \( M \); (3') \( r'(M) > 0 \) for some \( M \);
(4) \( \hat{R} \) is \( F \)-rational;
It remains to show (1) ⇔ $F$ is turn implies statements (1), (1'), (2), (2'), and (3). A common parameter test element for (2), (2'), (3), or (3') will imply (4).

The implication (4) ⇒ (5) is straightforward to see that (3') implies $R$ is an $F$-rational domain, which in turn implies statements (1), (1'), (2), (2'), and (3).

Now it remains to prove (4) ⇒ (3') in order to complete the proof. So we assume $\hat{R}$ is $F$-rational. Thus $R$ is Cohen-Macaulay and, by Remark 4.1, $r'_R(M) = 0$, for every finitely generated $R$-module $M$ (see [HY, Theorem 4.1]). Thus any single one of (1), (1'), (2), (2'), (3), or (3') will imply (4).

It is straightforward to see that (3') implies $R$ is an $F$-rational domain, which in turn implies statements (1), (1'), (2), (2'), and (3).

Now it remains to prove (4) ⇒ (3') in order to complete the proof. So we assume $\hat{R}$ is $F$-rational. Thus $R$ is Cohen-Macaulay and, by Remark 4.1, $r'_R(M) = 0$, for every finitely generated $R$-module $M$. As a result, we may assume $R$ is a complete $F$-rational domain (hence Cohen-Macaulay). But now it is enough to show $r'(R) > 0$ as $r'(M) = \text{rank}_R(M)r'(R)$ where $\text{rank}_R(M)$ denotes the torsion-free rank of $M$ over $R$.

One of the ingredients in this proof is the following Fact, which is explicitly stated in [HY, Theorem 3.3].

**Fact.** Let $(R, m)$ be a complete local domain of prime characteristic $p$. There exist $c \in R^*, c' \in R^*$ and $q'' = p^{c''}$ such that, for any $R$-modules $L$ with $\lambda(L) < \infty$, if $q'''$ is a test exponent for $cc'$ and $0 \subseteq L$, then $e_{HK}(L, R) - e_{HK}(L/K, R) \geq 1/(q''q''')^{\dim(R)}$ for every $K$ such that $0^*_L \not\supseteq K \subseteq L$.

Now we come back to the proof of Theorem 4.2. Remember that $\hat{R}$ is a complete $F$-rational local ring (hence a Cohen-Macaulay domain) now. Choose $c, c' \in R^*$ as in the above Fact. In [Ab], Aberbach has shown that $0^*_L = 0$ for every $\hat{R}$-module $L$ such that $\text{ppd}_R(L) < \infty$. Also, from Corollary 3.3, there is a test exponent, say $q''$, for $cc'$ and all $\hat{R}$-modules $L$ such that $\text{ppd}_R(L) < \infty$ and $\lambda(L) < \infty$, which forces $e_{\hat{HK}}(L, \hat{R}) - e_{\hat{HK}}(L/K, \hat{R}) \geq 1/(q''q''')^{\dim(\hat{R})}$ for every $0 \neq K \subseteq L$ by the above Fact. Now it follows from Definition 0.8 that $r'(R) \geq 1/(q''q''')^{\dim(R)} > 0$. □

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