Math. 632. Homework 1 (Part I)

1. Let $A$ be a finitely generated algebra over a field $k$. Prove that the set of maximal ideals of $A$ is dense in the Zariski topology on $\text{Spec}A$. Give an example of a ring $A$ where this is not true.

2. Give an example of a sheaf of ring on a topological space $X$ such that for any point $x \in X$ the ring of germs at $x$ is not a local ring.

3. Let $X = \mathbb{R}^n$ with usual topology defined by the Euclidean distance. For any open subset $U$ define $\mathcal{O}_X(U)$ to be ring of functions of class $C^k$.
   (i) Show that this defines a sheaf of rings and the pair $(X, \mathcal{O}_X)$ is a geometric space.
   (ii) Show that one can define an $n$-dimensional manifold of class $C^k$ as a geometric space locally isomorphic to $(X, \mathcal{O}_X)$.
   (iii) Show that maps of manifolds of class $C^k$ are exactly morphisms of the corresponding geometric spaces.

4. Describe explicitly the fibres of the morphism $f : \text{Spec}B \to \text{Spec}A$, where $A \to B$ is one of the following homomorphisms of rings $\phi : A \to B$:
   (i) $A = \mathbb{Z}$, $B = \mathbb{Z}[\sqrt{-1}]$, the ring of Gaussian integers, $\phi$ is the natural inclusion of rings.
   (ii) $A = \mathbb{C}[u,v], B = \mathbb{C}[x,y,z,u,v]/(x^2 + uy^2 + v)$ and $\phi$ is defined by sending $u, v$ to the cosets of $u, v$ modulo the ideal $(x^2 + uy^2 + v)$.

5. Let $\mathbb{G}_a = \text{Spec} k[t]$ (resp. $\mathbb{G}_m = \text{Spec} k[t, t^{-1}]$) be the additive group (resp. the multiplicative group scheme) over a field $k$ of characteristic $p > 0$. Show that the homomorphism of rings $k[t] \to k[t]$ (resp. $k[t, t^{-1}] \to k[t, t^{-1}]$), defined by the formula $t \mapsto t^p$ is a homomorphism of group schemes. Describe its kernel.

6. Prove that a scheme is separated if and only if the intersection of any affine subsets is affine. Show that any affine scheme is separated.