Suppose we have an autonomous system of two ODE’s:

\[
\begin{align*}
x' &= f(x, y) \\
y' &= g(x, y).
\end{align*}
\]  

(1)

We can write this as

\[
\begin{align*}
\frac{dx}{dt} &= f(x, y) \\
\frac{dy}{dt} &= g(x, y).
\end{align*}
\]  

(2)

But in (2), it looks like we can divide one equation by the other. For example, we may try to define the second equation by the first one and get

\[
\frac{dy}{dx} = \frac{g(x, y)}{f(x, y)}.
\]  

(3)

(This can be made precise using the chain rule of differentiation.) But we can then treat this as a single first order ODE, with \( y \) regarded as a function of \( x \), so we have converted the problem (1) to the simpler problem (3).

In order for this to work, we must have

\[
\frac{dx}{dt} \neq 0,
\]

which means

\[
f(x, y) \neq 0.
\]

Alternately, if \( g(x, y) \neq 0 \), we can proceed analogously, dividing the first equation (2) by the second equation. However, what if

\[
f(x, y) = g(x, y) = 0?
\]

Then at the point \((x, y)\), we have an equilibrium (=constant) solution of the system (1). So, we understand those points as well.

**Example 1:**

\[
\begin{align*}
y' &= xy^2 \\
x' &= x^2 y.
\end{align*}
\]  

(4)

Solution: Write the system in the form

\[
\frac{dy}{dx} = \frac{y}{x},
\]

which gives

\[
y = Kx.
\]  

(5)

From the second equation (4), we then get

\[
x' = Kx^3,
\]

(1)
which gives
\[-\frac{1}{2x^2} = Kt + C\]
or
\[x = \pm \sqrt{\frac{1}{D - 2Kt}}.\]
which then gives
\[y = \pm \frac{K}{\sqrt{D - 2Kt}}\]
(same sign). Note carefully that we are not allowed to absorb the constant $K$ into other constants when solving for $x$, because the constant occurs in (5).

**Example 2:** The autonomous second order ODE:

(6) \[y'' = f(y, y').\]

We rewrite (6) as a first order system

(7) \[
\begin{align*}
y' &= x \\
x' &= f(y, x).
\end{align*}
\]

So we have
\[
\frac{dx}{dy} = \frac{f(y, x)}{x}
\]
which we can solve to find $x$ in terms of $y$, then plug into the first equation (7) to find $y$ in terms of $t$.

**Example 3:** Using the general method of the previous example, solve:

\[y'' = \frac{(y')^2}{y}.\]

Solution: rewrite the equation as the system

(8) \[
\begin{align*}
y' &= x \\
x' &= \frac{x^3}{y}.
\end{align*}
\]

This gives
\[
\frac{dx}{dy} = \frac{x}{y'}
\]
or
\[x = Ky.\]

From the first equation (8), we then find
\[y' = Ky,
\]
which gives
\[y = Ce^{Kt}.\]