1. Observed Order of Accuracy
I. Consider $u(x) = e^x$, $x_0 = 0$. Compute $D_+ u(x_0)$, $D_0 u(x_0)$, $D_3 u(x_0)$, using $h = 0.1, 0.05, 0.025, 0.0125$. Compute the error. For each finite difference approximation, construct a table of: $h$, error, error/$h$, error/$h^2$ and error/$h^3$, confirm the expected order of accuracy. Compute the constants $C$ and compare with theoretical values. Plot $\log(error)$ against $\log h$.

II. Perform Richardson’s extrapolation on the values of $D_+ u(x_0)$, and $D_0 u(x_0)$.

2. Roundoff Errors
Consider again $u(x) = e^x$, $x_0 = 0$. Compute $D_+ u(x_0)$ for a sequence of increasingly small $h$ values, for example, $h = 10^{-n}$. Plot the error against $h$ and explain your observations.

3. Finite Differences using interpolating polynomials
Let $p(x)$ be the interpolating polynomial to $u(x)$ at the points $x_0$, $x_0 - h$ and $x_0 - 2h$. Show that $p'(x_0)$ gives exactly the 3-point one-sided difference approximation $D_2(x_0)$ derived in class.

4. High order approximations using Richardson’s extrapolation
Consider the backward difference approximation $D_- u(x_0)$ based on a step size $h$, and on a step size $2h$. Show that Richardson’s extrapolation on the two approximations gives $D_2(x_0)$.

5. Finite difference approximations with unequally spaced points.
I. Use the method of undetermined coefficients to derive the 3-point difference approximation to $u''(x_0)$ using the unequally spaced points $u(x_0 - h_1)$, $u(x_0)$ and $u(x_0 + h_2)$.

$$u''(x_0) \approx \frac{2u(x_0 - h_1)}{h_1(h_1 + h_2)} - \frac{2u(x_0)}{h_1 h_2} + \frac{2u(x_0 + h_2)}{h_2(h_1 + h_2)}$$

Comment on the accuracy of the approximation.

II. Use this formula to approximate the derivative of $e^x$ at $x_0 = 0$ when

a. $h_1 = 0.05$ and $h_2 = 0.1$

b. $h_1 = 0.005$ and $h_2 = 0.01$

c. $h_1 = 0.0005$ and $h_2 = 0.001$

Compare with the exact answer. By what factor is the error decreasing at each step?

III. Confirm that the approximation reduces to the centered formula when $h_1 = h_2 = h$ and use it to compute the second derivative with (a) $h = 0.1$ (b) $h = 0.01$ and (c) $h = 0.001$. By what factor is the error decreasing at each step?

IV. Discuss the accuracy of centered vs. non-centered approximations.