1. Consider the linear system

\[
\begin{pmatrix}
2 & -1 \\
-1 & 2 \\
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
\end{pmatrix}
= 
\begin{pmatrix}
1 \\
1 \\
\end{pmatrix}.
\]

I. Set up the iteration matrices, \(G_J\) and \(G_{GS}\). Compute \(\rho(G)\) and \(\|G\|_2\).

II. Perform 3 iterations by hand for each method. Compute the error in each solution component. Compute the ratio \(\|\epsilon^{(n+1)}\|_\infty / \|\epsilon^{(n)}\|_\infty\). (Set up a table with \(n\), \(x_1^{(n)}\), \(x_2^{(n)}\), \(e_1^{(n)}\), \(e_2^{(n)}\), \(\|\epsilon^{(n)}\|_\infty\) and \(\|\epsilon^{(n+1)}\|_\infty / \|\epsilon^{(n)}\|_\infty\)). Discuss the rate of convergence.

III. Set up the iteration matrix for SOR. Show that if \(\rho(G) < 1\), then \(0 < \omega < 2\).

IV. Compute \(\rho(G)\) as a function of \(\omega\) (You may use Matlab to compute \(\max|\lambda_k(\omega)|\)). Plot \(\rho(G)\) vs. \(\omega\) and find \(\omega_{opt}\). Compare with the theoretical prediction.

V. Perform 3 SOR iterations by hand using \(\omega_{opt}\). Compute the error. Compute the ratio \(\|\epsilon^{(n+1)}\|_\infty / \|\epsilon^{(n)}\|_\infty\). (Set up a table with \(n\), \(x_1^{(n)}\), \(x_2^{(n)}\), \(e_1^{(n)}\), \(e_2^{(n)}\), \(\|\epsilon^{(n)}\|_\infty\) and \(\|\epsilon^{(n+1)}\|_\infty / \|\epsilon^{(n)}\|_\infty\)). Discuss the rate of convergence.

2. Show that if \(A\) is strictly diagonally dominant, \(\|G_J\| < 1\) (in which norm?), and therefore the iteration converges.

3. Aitken’s acceleration. Assume that the iteration matrix \(G\) has a complete set of e-vectors, \(r_k\), with corresponding e-values \(\lambda_k\). Assume that \(\lambda_1\) is the largest in magnitude. The set of e-vectors can be used as a basis.

I. By expanding the initial error \(\epsilon^{(0)}\) in terms of the basis \(r_k\), show that \(\epsilon^{(n+1)} \approx \lambda_1 \epsilon^{(n)}\).

II. From the above, one has

\[
\begin{align*}
\epsilon^{(n+1)} & \approx \lambda_1 \epsilon^{(n)} \\
\epsilon^{(n)} & \approx \lambda_1 \epsilon^{(n-1)}.
\end{align*}
\]

Eliminate \(\lambda_1\) between the above two (approximate) equations. Replace \(\epsilon^{(n)} = x - x^{(n)}\) etc. and rearrange the resulting equations to show that the exact solution \(x = (x_1, ..., x_n)^T\) satisfies
\[ x_i \approx \frac{x_i^{(n+1)} x_i^{(n-1)} - (x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}} \approx x_i^{(n+1)} - \frac{(x_i^{(n+1)} - x_i^{(n)})^2}{x_i^{(n+1)} - 2x_i^{(n)} + x_i^{(n-1)}} \]

III. Consider the linear system

\[
\begin{pmatrix}
-2.2 & 1 & 1 \\
0.8 & -2.2 & 1 \\
1 & 0.9 & -2.1
\end{pmatrix}
\begin{pmatrix}
x_1 \\
x_2 \\
x_3
\end{pmatrix} =
\begin{pmatrix}
0.2 \\
0.4 \\
0.2
\end{pmatrix}.
\]

Start with an initial guess \( x^{(0)} = (0, 0, 0)^T \) and perform 3 Gauss-Seidel iterations. Use Aitken’s acceleration on these three iterates to obtain an improved approximations.

IV. Write a small routine for Gauss-Seidel’s method and check how many GS iterations would be required to achieve the same accuracy.